Homogenization of parabolic equations with a continuum of space and time scales.

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Abstract

This paper addresses the issue of homogenization of linear divergence form parabolic operators in situations where no ergodicity and no scale separation in time or space are available. Namely, we consider divergence form linear parabolic operators in $\Omega \subset \mathbb{R}^n$ with $L^\infty(\Omega \times (0,T))$-coefficients. It appears that the inverse operator maps the unit ball of $L^2(\Omega \times (0,T))$ into a space of functions which at small (time and space) scales are close in H^1-norm to a functional space of dimension n. It follows that once one has solved these equations at least n-times it is possible to homogenize them both in space and in time, reducing the number of operations counts necessary to obtain further solutions. In practice we show that under a Cordes type condition that the first order time derivatives and second order space derivatives of the solution of these operators with respect to harmonic coordinates are in L^2 (instead of H^{-1} with Euclidean coordinates). If the medium is time independent then it is sufficient to solve n times the associated elliptic equation in order to homogenize the parabolic equation.

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1 Introduction and main results

Let Ω be a bounded and convex domain of class C^2 of $\mathbb{R}^n$. Let $T > 0$. Consider the following parabolic PDE

$$\begin{cases} \partial_t u = \operatorname{div}\big(a(x,t)\nabla u(x,t)\big) + g \quad \text{in} \quad \Omega \times (0,T) \\ u(x,t) = 0 \quad \text{for} \quad (x,t) \in \big(\partial\Omega \times (0,T)\big) \cup \big(\Omega \cup \{t=0\}\big). \end{cases} \tag{1.1}$$

Write $\Omega_T := \Omega \times (0,T)$. g is a function in $L^2(\Omega_T)$. $(x,t) \to a(x,t)$ is a mapping from Ω_T into the space of symmetric positive definite matrices with entries in $L^\infty(\Omega_T)$. Assume a to be uniformly elliptic on the closure of Ω_T. This paper addresses the issue of the homogenization of (1.1) in space and time in situations where scale separation and ergodicity at small scales are not available (see [15], [43] and [3] for an introduction to classical homogenization theory). For that purpose, we will introduce in subsection 1.1 theorems establishing under Cordes type conditions the increase of regularity of solutions of (1.1) when derivatives are taken with respect to harmonic coordinates instead of Euclidean coordinates. In subsections 1.2, 1.3 these results will be used to homogenize (1.1) in space and in time. More precisely, assume a to be written on a fine tessellation with N degrees of freedom. If a is time independent, then by solving n-times an elliptic boundary value-problem associated to (1.1) (at a cost of $O(N(\ln N)^{n+3})$ operations using the Hierarchical matrix method [11]) it is possible approximate the solutions of (1.1) by solving an homogenized operator with N^α degrees of freedom ($\alpha < 1$, $\alpha = 0.2$ for instance) or with a fixed number M of degrees of freedom (numerical experiments given at the end of this paper have been conducted with $N = 16641$ and $M = 9$), this problem is of practical importance for oil extraction and reservoir modeling in geophysics. If a is also characterized by a continuum of time scales, then the method presented here does not reduce the number of operation counts necessary to solve (1.1) only one time. However if one needs to solve (1.1) K ($K > n$) times (with different right hand sides) then by solving (1.1) n times it is possible to obtain an approximation of the solutions of (1.1) by solving an homogenized (in space and time) parabolic equation written on a coarse tessellation with coarse time steps.

1.1 Compensation phenomenon

Let F be the solution of the following parabolic equation

$$\begin{cases} \partial_t F = \operatorname{div}\big(a(x,t)\nabla F(x,t)\big) \quad \text{in} \quad \Omega_T \\ F(x,t) = x \quad \text{for} \quad (x,t) \in \big(\partial\Omega \times (0,T)\big) \\ \operatorname{div}\big(a(x,0)\nabla F(x,0)\big) = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

By (1.2) we mean that $F := (F_1, \dots, F_n)$ is a n-dimensional vector field such that each of its entries satisfies

$$\begin{cases} \partial_t F_i = \operatorname{div}\big(a(x,t)\nabla F_i(x,t)\big) \quad \text{in} \quad \Omega_T \\ F_i(x,t) = x_i \quad \text{for} \quad (x,t) \in \big(\partial\Omega \times (0,T)\big) \\ \operatorname{div}\big(a(x,0)\nabla F_i(x,0)\big) = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

Observe that if a is time independent then F is the solution of an elliptic boundary value problem.

Definition 1.1. Write

$$\sigma := {}^t\nabla F a \nabla F. \tag{1.4}$$

Write β_σ the Cordes parameter associated to σ defined by

$$\beta_\sigma := \operatorname{esssup}_{(x,t)\in\Omega_T}\Big(n - \frac{\big(\operatorname{Trace}[\sigma]\big)^2}{\operatorname{Trace}[{}^t\sigma\sigma]}\Big). \tag{1.5}$$

Observe that since

$$\beta_\sigma = \operatorname{esssup}_{(x,t)\in\Omega_T}\Big(n - \frac{\big(\sum_{i=1}^n \lambda_{i,\sigma(x,t)}\big)^2}{\sum_{i=1}^n \lambda_{i,\sigma(x,t)}^2}\Big). \tag{1.6}$$

where $(\lambda_{i,M})$ denotes the eigenvalues of M, β_σ is a measure of the anisotropy of σ.

1.1.1 Time independent medium.

In this subsection we assume that a does not depend on time t. Write for $p \geq 2$, $W_D^{2,p}$ (D for Dirichlet boundary condition) the Banach space $W_D^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Equip $W_D^{2,p}(\Omega)$ with the norm

$$\|v\|^2_{W_D^{2,p}(\Omega)} := \int_\Omega \big(\sum_{i,j}(\partial_i\partial_j v)^2\big)^{\frac{p}{2}}. \tag{1.7}$$

Equip the space $L^p(0,T,W_D^{2,p}(\Omega))$ with the norm

$$\|v\|^p_{L^p(0,T,W_D^{2,p}(\Omega))} = \int_0^T \int_\Omega \big(\sum_{i,j}(\partial_i\partial_j v)^2\big)^{\frac{p}{2}}\, dx\, dt. \tag{1.8}$$

Theorem 1.1. *Assume that $\partial_t a \equiv 0$, $g \in L^2(\Omega_T)$, Ω is convex, $\beta_\sigma < 1$ and $(\operatorname{Trace}[\sigma])^{\frac{n}{4}-1} \in L^\infty(\Omega)$ then $u \circ F^{-1} \in L^2(0,T,W_D^{2,2}(\Omega))$ and*

$$\|u \circ F^{-1}\|_{L^2(0,T,W_D^{2,2}(\Omega))} \leq \frac{C}{1-\beta_\sigma^{\frac{1}{2}}}\|g\|_{L^2(\Omega_T)}. \tag{1.9}$$

Remark 1.1. The constant C can be written

$$C = \frac{C_n}{(\lambda_{\min}(a))^{\frac{n}{4}}} \left\| (\mathrm{Trace}[\sigma])^{\frac{n}{4}-1} \right\|_{L^\infty(\Omega)}.$$

Through this paper, we write

$$\lambda_{\min}(a) := \inf_{(x,t)\in\Omega_T} \inf_{l\in\mathbb{R}^n, |l|=1} {}^t l.a(x,t).l. \tag{1.10}$$

Remark 1.2. According to theorem 1.1 although the second order derivatives of u with respect to Euclidean coordinates are only in $L^2(0,T,H^{-1}(\Omega))$, they are in $L^2(\Omega_T)$ with respect to harmonic coordinates.

Remark 1.3. Observe that if a is time independent then F and σ are time independent and F is the solution of the following elliptic problem:

$$\begin{cases} \operatorname{div} a\nabla F = 0 \quad \text{in} \quad \Omega \\ F(x) = x \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.11}$$

In dimension one F is trivially an homeomorphism. In dimension 2 this property follows from topological constraints [2] (even with $a_{i,j} \in L^\infty(\Omega)$), [6] (one can also deduce from [6] that for $n = 2$, if a is smooth then the conditions $\beta_\sigma < 1$ and $(\mathrm{Trace}[\sigma])^{-1} \in L^\infty(\Omega)$ are satisfied). In dimension three and higher F can be non-bijective even if a is smooth, we refer to [6] and [21], however in dimension 3 the assumption $(\mathrm{Trace}[\sigma])^{\frac{n}{4}-1} \in L^\infty(\Omega_T)$ implies that F is an homeomorphism. If $n \geq 4$ we need to assume that F is an homeomorphism to prove the theorem.

Remark 1.4. In fact the condition $(\mathrm{Trace}(\sigma))^{-1} \in L^p(\Omega_T)$ for $p < \infty$ depending on n is sufficient to obtain theorem 1.1 and the following compensation theorems. For the sake of clarity this paper has been restricted to $(\mathrm{Trace}(\sigma))^{-1} \in L^\infty(\Omega_T)$.

Remark 1.5. Write

$$\mu_\sigma := \mathrm{esssup}_{\Omega_T} \frac{\lambda_{\max}(\sigma)}{\lambda_{\min}(\sigma)}. \tag{1.12}$$

It is easy to check that μ_σ is bounded by an increasing function of $(1-\beta_\sigma)^{-1}$ and in dimension two $\beta_\sigma < 1$ is equivalent to $\mu_\sigma < \infty$.

Remark 1.6. Theorem 1.1 has been called compensation phenomenon because the composition by F^{-1} increases the regularity of $u \in L^2(0,T,H_0^1(\Omega))$. The choice of this name has been motivated by F. Murat and L. Tartar's work on H-convergence [53] which is also based on on a regularization property called compensated compactness or div-curl lemma introduced in the 70's by Murat and Tartar [52], [67] (we also refer to [25] for refinements of the div-curl lemma).

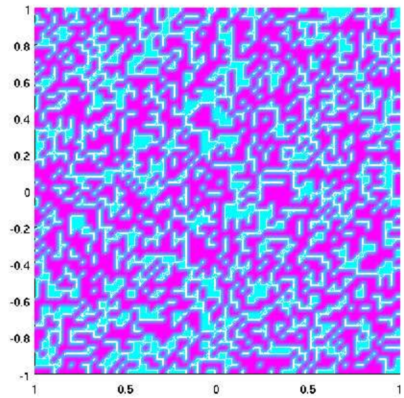
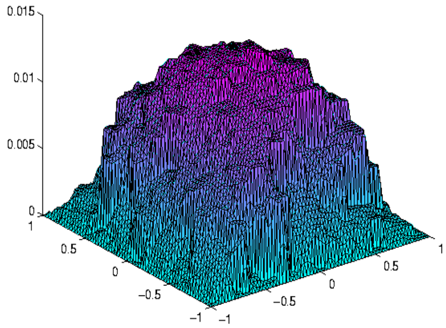
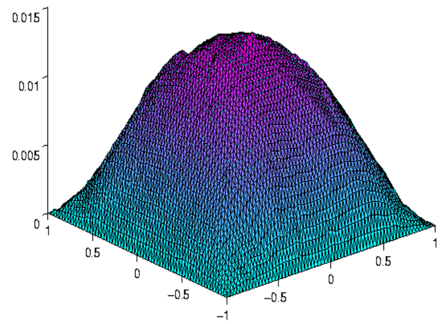


Figure 1: Site Percolation

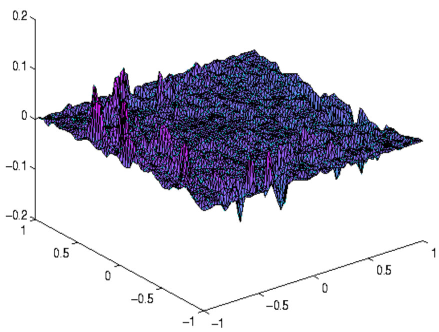
The compensation phenomena presented in this subsection can be observed numerically. In figure 1, the value of a is set to be equal to 1 or 100 with probability 1/2 on each triangle of a fine mesh characterized by 16641 nodes and 32768 triangles. (1.1) has been solved numerically on that mesh with $g = 1$. u, $u \circ F^{-1}$, $\partial_x u$ and $\partial x(u \circ F^{-1})$ have been plotted at time $t = 1$ in figure 2.



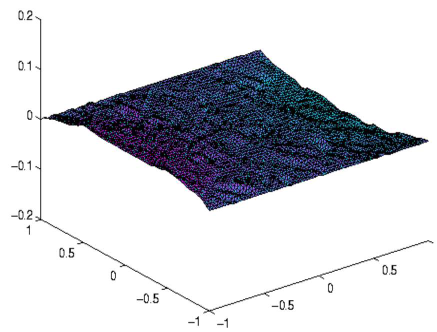
(a) u**.**



(b) $u \circ F^{-1}$**.**



(c) $\partial_x u$**.**



(d) $\partial x(u \circ F^{-1})$**.**

Figure 2: u, $u \circ F^{-1}$, $\partial_x u$ and $\partial x(u \circ F^{-1})$ at time $t = 1$ for the time independent site percolating medium.

In situations where $g \in L^\infty(0,T,L^2(\Omega))$, $\partial_t g \in L^2(0,T,H^{-1}(\Omega))$ or $g \in L^p(\Omega_T)$ with $p > 2$, one can obtain a higher regularity for $u \circ F^{-1}$. This is the object of the following theorems.

Theorem 1.2. *Assume that Ω is convex, $g \in L^\infty(0,T,L^2(\Omega))$, $\partial_t g \in L^2(0,T,H^{-1}(\Omega))$, $\partial_t a \equiv 0$, $\beta_\sigma < 1$ and $(\mathrm{Trace}[\sigma])^{\frac{n}{4}-1} \in L^\infty(\Omega_T)$ then for all $t \in [0,T]$, $u \circ F^{-1}(.,t) \in W^{2,2}_D(\Omega)$ and*

$$\|u \circ F^{-1}(.,t)\|_{W^{2,2}_D(\Omega)} \leq \frac{C}{1-\beta_\sigma^{\frac{1}{2}}}\Big(\|g\|_{L^\infty(0,T,L^2(\Omega))} + \|\partial_t g\|_{L^2(0,T,H^{-1}(\Omega))}\Big). \tag{1.13}$$

Remark 1.7. The constant C can be written

$$C = \frac{C_{n,\Omega}}{(\lambda_{\min}(a))^{\frac{n}{4}}}\big\|(\mathrm{Trace}[\sigma])^{\frac{n}{4}-1}\big\|_{L^\infty(\Omega_T)}(1+\frac{1}{\lambda_{\min}(a)})^{\frac{1}{2}}.$$

Theorem 1.3. *Assume that Ω is convex, $g(.,0) \in L^2(\Omega)$, $\partial_t g \in L^2(0,T,H^{-1}(\Omega))$, $g \in L^p(\Omega_T)$, $\partial_t a \equiv 0$, $\beta_\sigma < 1$ and $(\mathrm{Trace}[\sigma])^{\frac{n}{4}-1} \in L^\infty(\Omega_T)$ then there exists a real number $p_0 > 2$ depending only on n, Ω and β_σ such that for each p such that $2 \leq p < p_0$ one has*

$$\begin{aligned}\|u \circ F^{-1}\|_{L^p(0,T,W^{2,p}_D(\Omega))} \leq & \frac{C}{1-\beta_\sigma^{\frac{1}{2}}}\big(\|g\|_{L^p(\Omega_T)} \\ & + \big\|g(.,0)\big\|_{L^2(\Omega)} + \|\partial_t g\|_{L^2(0,T,H^{-1}(\Omega))}\big).\end{aligned} \tag{1.14}$$

Remark 1.8. The constant C can written

$$C = \frac{C_{n,\Omega,p}}{(\lambda_{\min}(a))^{\frac{n}{4}}}\big\|(\mathrm{Trace}[\sigma])^{\frac{n}{4}-1}\big\|_{L^\infty(\Omega_T)}\big(1+\frac{1}{\lambda_{\min}(a)}\big)^{\frac{1}{2}}.$$

Write

$$\|v\|_{C^\gamma(\Omega)} := \sup_{x,y\in\Omega, x\neq y}\frac{|v(x)-v(y)|}{|x-y|^\gamma}. \tag{1.15}$$

Theorem 1.4. *Assume that $n \leq 2$, Ω is convex, $g(.,0) \in L^2(\Omega)$, $\partial_t g \in L^2(0,T,H^{-1}(\Omega))$, $g \in L^p(\Omega_T)$, $\partial_t a \equiv 0$, $\beta_\sigma < 1$, $(\mathrm{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)$ and $g \in L^2[0,T;L^{p^*}(\Omega)]$ with $2 < p^*$. Then there exists $p \in (2,p^*]$ and $\gamma(p) > 0$ such that*

$$\begin{aligned}\Big(\int_0^T \big\|\nabla(u\circ F^{-1})(.,t)\big\|^2_{C^\gamma(\Omega)}\,dt\Big)^{\frac{1}{2}} \leq & \frac{C}{1-\beta_\sigma^{\frac{1}{2}}}\big(\|g\|_{L^p(\Omega_T)} \\ & + \big\|g(.,0)\big\|_{L^2(\Omega)} + \|\partial_t g\|_{L^2(0,T,H^{-1}(\Omega))}\big).\end{aligned} \tag{1.16}$$

Remark 1.9. The constant C in (1.16) depends on n, p, Ω, $\lambda_{\min}(a)$ and $\big\|(\mathrm{Trace}(\sigma))^{-1}\big\|_{L^\infty(\Omega_T)}$. It is easy to check that if $n=1$ then the theorem is valid with $\gamma = 1/2$.

In the following theorems Ω is not assumed to be convex.

Theorem 1.5. *Assume $n \geq 2$ and $\partial_t a \equiv 0$. Let $p > 2$. There exist a constant $C^* = C^*(n, \partial\Omega) > 0$ a real number $\gamma > 0$ depending only on n, Ω and p such that if $\beta_\sigma < C^*$ then*

$$\Big(\int_0^T \big\|\nabla(u \circ F^{-1})(.,t)\big\|^2_{C^\gamma(\Omega)}\,dt\Big)^{\frac{1}{2}} \leq C\big(\|g\|_{L^p(\Omega_T)} + \big\|g(.,0)\big\|_{L^2(\Omega)} + \|\partial_t g\|_{L^2(0,T,H^{-1}(\Omega))}\big). \tag{1.17}$$

Remark 1.10. The constant C in (1.17) depends on n, γ, Ω, C^*, $\lambda_{\min}(a)$ and $\big\|(\mathrm{Trace}(\sigma))^{\frac{n}{2p}-1}\big\|_{L^\infty(\Omega_T)}$.

It is easy to check that if $a = e(x)S(x,t)$ where e is a time independent symmetric uniformly elliptic matrix with $L^\infty(\Omega)$ entries and S is a regular uniformly positive function then the results given in this sub-section and the homogenization schemes of sub-section 1.2 remain valid with the time independent harmonic coordinates associated to e, i.e. solution of $-\operatorname{div} e\nabla F = 0$.

1.1.2 Medium with a continuum of time scales.

In this subsection the entries of a are merely in $L^\infty(\Omega_T)$. We need to introduce the following Cordes type condition.

Condition 1.1. *We say that condition 1.1 is satisfied if and only if there exists $\delta \in (0,\infty)$ and $\epsilon > 0$ such that*

$$\mathrm{esssup}_{\Omega_T} \frac{\delta^2\,\mathrm{Trace}[{}^t\sigma\sigma] + 1}{\big(\delta\,\mathrm{Trace}[\sigma] + 1\big)^2} \leq \frac{1}{n+\epsilon}. \tag{1.18}$$

Write

$$z_\sigma := \mathrm{esssup}_{\Omega_T} n \frac{\mathrm{Trace}[{}^t\sigma\sigma]}{(\mathrm{Trace}[\sigma])^2}. \tag{1.19}$$

Observe that z_σ is a measure of anisotropy of σ, in particular $1 \leq z_\sigma \leq n$ and $z_\sigma = 1$ if σ is isotropic. Write

$$y_\sigma := \|\,\mathrm{Trace}[\sigma]\|_{L^\infty(\Omega_T)} \big\|(\mathrm{Trace}[\sigma])^{-1}\big\|_{L^\infty(\Omega_T)}. \tag{1.20}$$

Proposition 1.1. *If $\|\,\mathrm{Trace}[\sigma]\|_{L^\infty(\Omega_T)} < \infty$ and $\big\|(\mathrm{Trace}[\sigma])^{-1}\big\|_{L^\infty(\Omega_T)} < \infty$ then condition 1.1 is satisfied with*

$$\delta := n\big\|(\mathrm{Trace}[\sigma])^{-1}\big\|_{L^\infty(\Omega_T)} \tag{1.21}$$

and with $\epsilon := \frac{2ny_\sigma - n}{2ny_\sigma^2}$ provided that $z_\sigma \leq 1 + \frac{\epsilon}{n}$.

Remark 1.11. Observe that in dimension one $z_\sigma = 1$, thus for $n = 1$ condition 1.1 is satisfied is $\mathrm{Trace}[\sigma] \in L^\infty(\Omega_T)$ and $(\mathrm{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)$.

We have the following theorems

Theorem 1.6. *Assume that Ω is convex, and condition 1.1 is satisfied then $u \circ F^{-1} \in L^2\big(0, T, W_D^{2,2}(\Omega)\big)$, $\partial_t(u \circ F^{-1}) \in L^2(\Omega_T)$ and*

$$\|u \circ F^{-1}\|_{L^2(0,T,W_D^{2,2}(\Omega))} + \|\partial_t(u \circ F^{-1})\|_{L^2(\Omega_T)} \leq C\|g\|_{L^2(\Omega_T)} \tag{1.22}$$

where C depends on Ω, n, δ and ϵ.

Remark 1.12. According to theorem 1.6 although the second order space derivatives and first order time derivatives of u with respect to Euclidean coordinates are only in $L^2(0, T, H^{-1}(\Omega))$, they are in $L^2(\Omega_T)$ with respect to harmonic coordinates.

Similarly we obtain the following theorems in situations where $g \in L^p(\Omega_T)$ with $p > 2$.

Theorem 1.7. *Assume that Ω is convex, and condition 1.1 is satisfied then there exists a number $p_0 > 2$ depending on n, Ω, ϵ such that for $p \in (2, p_0)$, $u \circ F^{-1} \in L^p\big(0, T, W_D^{2,p}(\Omega)\big)$, $\partial_t(u \circ F^{-1}) \in L^p(\Omega_T)$ and*

$$\|u \circ F^{-1}\|_{L^p(0,T,W_D^{2,p}(\Omega))} + \|\partial_t(u \circ F^{-1})\|_{L^p(\Omega_T)} \leq C\|g\|_{L^p(\Omega_T)} \tag{1.23}$$

where C depends on Ω, n, δ and ϵ.

Theorem 1.8. *Assume that Ω is convex, and condition 1.1 is satisfied then there exists a number $\alpha_0 > 2$ depending on n, Ω, ϵ such that for $\alpha \in (0, \alpha_0)$, $\nabla(u \circ F^{-1}) \in L^2(0, T, C^\alpha(\Omega))$ and*

$$\|\nabla(u \circ F^{-1})(., t)\|_{L^2(0,T,C^\alpha(\Omega))} \leq C\|g\|_{L^p(\Omega_T)} \tag{1.24}$$

where C depends on Ω, δ, n, and ϵ.

These compensation phenomena can be observed numerically. We consider in dimension $n = 2$,

$$a(x, y, t) = \frac{1}{6}\Big(\sum_{i=1}^{5} \frac{1.1 + \sin(2\pi x'/\epsilon_i)}{1.1 + \sin(2\pi y'/\epsilon_i)} + \sin(4x'^2 y'^2) + 1\Big) \tag{1.25}$$

with $x' = x + \sqrt{2}t$, $y' = y - \sqrt{2}t$, $\epsilon_1 = \frac{1}{5}$, $\epsilon_2 = \frac{1}{13}$, $\epsilon_3 = \frac{1}{17}$, $\epsilon_4 = \frac{1}{31}$ and $\epsilon_5 = \frac{1}{65}$. This medium has been plotted in figure 3 at time 0 (observe that $\lambda_{\max}(a)/\lambda_{\min}(a) \sim 100$).

(1.1) has been solved numerically on that mesh with $g \equiv 1$ on the fine mesh characterized by 16641 nodes and 32768 triangles. Figure 4 shows $\partial_x u$ and $\partial_x(u \circ F^{-1})$ at time 0.3.

In figure 5 and 6, the value of x_0 is set to $x_0 := (0.75, -0.25)$ and the curves $t \to u(x_0, t), u \circ F^{-1}(x_0, t), \nabla u(x_0, t), \nabla u \circ F^{-1}(x_0, t)$ are plotted from $t = 0$ to $t = 0.3$

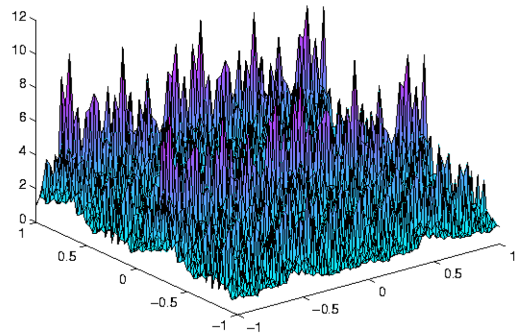
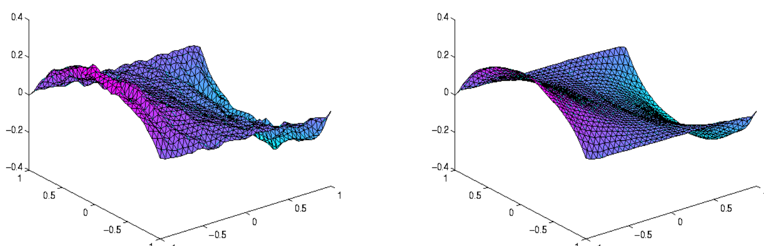
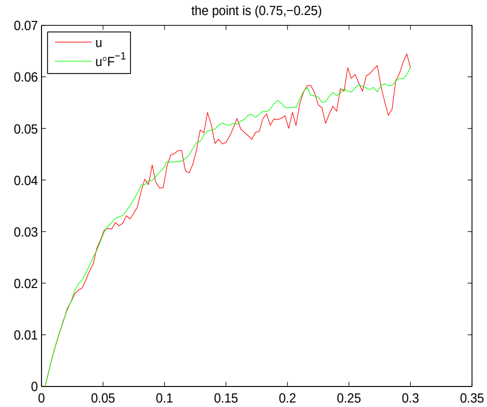


Figure 3: a at time 0.



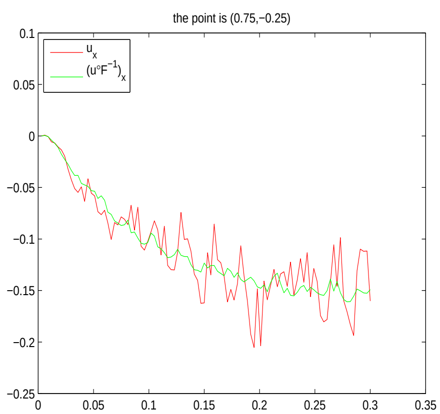
(a) $\partial_x u$**.** **(b)** $\partial_x(u \circ F^{-1})$**.**

Figure 4: $\partial_x u$ and $\partial_x(u \circ F^{-1})$ at time $t = 0.3$ for the Multi-scale Trigonometric time dependent Medium.

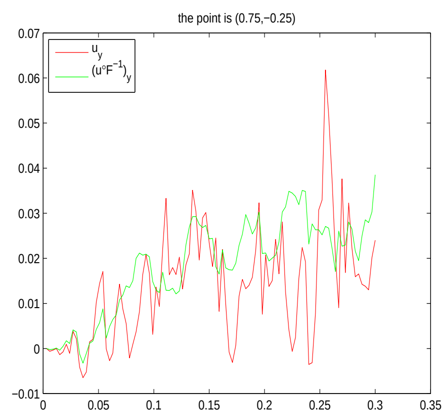


(a) u ***and*** $u \circ F^{-1}$**.**

Figure 5: $t \to u(x_0, t), u \circ F^{-1}(x_0, t)$ from $t = 0$ to $t = 0.3$ with $x_0 := (0.75, -0.25)$



(a) $\partial_x u$ ***and*** $\partial_x(u \circ F^{-1})$**.**



(b) $\partial_y u$ ***and*** $\partial_y(u \circ F^{-1})$**.**

Figure 6: $t \to \nabla u(x_0, t), \nabla u \circ F^{-1}(x_0, t)$ from $t = 0$ to $t = 0.3$ with $x_0 := (0.75, -0.25)$

1.2 Homogenization in space.

Let X_h be a finite dimensional subspace of $H^1_0(\Omega) \cap W^{1,\infty}(\Omega)$[1] with the following approximation property: there exists a constant C_X such that for all $f \in W^{2,2}_D(\Omega)$

$$\inf_{v\in X_h} \|f-v\|_{H^1_0(\Omega)} \leq C_X h \|f\|_{W^{2,2}_D(\Omega)}. \tag{1.26}$$

It is known and easy to check that the set of piecewise linear functions on a triangulation of Ω satisfies condition (1.26) provided that the length of the edges of the triangles are bounded by h (C_X in (1.26) being given by the aspect ratio of the triangles).

For media characterized by a continuum of time scales we will consider twice differentiable elements satisfying the following usual inverse inequalities (see section 1.7 of [29]): for $v \in X_h$,

$$\|v\|_{W^{2,2}_D(\Omega)} \leq C_X h^{-1} \|v\|_{H^1_0(\Omega)}. \tag{1.27}$$

and

$$\|v\|_{H^1_0(\Omega)} \leq C_X h^{-1} \|v\|_{L^2(\Omega)}. \tag{1.28}$$

In this paper we will use splines to ensure that condition (1.27) is satisfied (observe that it requires the quasi-uniformity of the (coarse) mesh, i.e. a bound on the aspect ratio of the (coarse) triangles).

For $t \in (0,T)$ let us define

$$V_h(t) := \big\{\varphi \circ F(x,t) \,:\, \varphi \in X_h\big\}. \tag{1.29}$$

Write $L^2\big(0,T;H^1_0(\Omega)\big)$ the usual Sobolev space associated to the norm

$$\|v\|^2_{L^2(0,T;H^1_0(\Omega))} := \int_0^T \big\|v(.,t)\big\|^2_{H^1_0(\Omega)}\,dt. \tag{1.30}$$

Write Y_T the subspace of $L^2\big(0,T;H^1_0(\Omega)\big)$ such that for each $v \in Y_T$ and $t \in [0,T]$, $x \to v(x,t)$ belongs to $V_h(t)$.
Write u_h the solution in Y_T of the following system of ordinary differential equations:

$$\begin{cases} (\psi, \partial_t u_h)_{L^2(\Omega)} + a[\psi, u_h] = (\psi, g)_{L^2(\Omega)} & \text{for all } t \in (0,T) \text{ and } \psi \in V_h(t) \\ u_h(x,0) = 0. \end{cases} \tag{1.31}$$

Write

$$a[v,w] := \int_\Omega {}^t\nabla v(x,t) a(x,t) \nabla w(x,t)\,dx. \tag{1.32}$$

[1] $W^{1,\infty}$ is the usual space of uniformly Lipschitz continuous functions.

1.2.1 Time independent domain

We have the following theorem

Theorem 1.9. *Assume that* $\partial_t a \equiv 0$, Ω *is convex,* $\beta_\sigma < 1$ *and* $(\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)$ *then*

$$\big\|(u-u_h)(.,T)\big\|_{L^2(\Omega)} + \big\|u-u_h\big\|_{L^2(0,T;H^1_0(\Omega))} \leq Ch\|g\|_{L^2(\Omega_T)}. \tag{1.33}$$

Remark 1.13. The constant C depends on C_X, n, Ω, $\lambda_{\min}(a)$ and $\big\|(\text{Trace}[\sigma])^{-1}\big\|_{L^\infty(\Omega_T)}$. If $n \geq 5$ it also depends on $\big\|\text{Trace}[\sigma]\big\|_{L^\infty(\Omega_T)}$ and if $n=1$ it also depends on $\lambda_{\max}(a)$.

1.2.2 Medium with a continuum of time scales.

Theorem 1.10. *Assume that* Ω *is convex, and condition 1.1 is satisfied then*

$$\big\|(u-u_h)(T)\big\|_{L^2(\Omega)} + \big\|u-u_h\big\|_{L^2(0,T,H^1_0(\Omega))} \leq Ch\|g\|_{L^2(\Omega_T)}. \tag{1.34}$$

Remark 1.14. The constant C depends on C_X, n, Ω, δ and ϵ, $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

The system of ordinary differential equations (1.31) is still characterized by a continuum of time scales in situations where the entries of a merely belong to $L^\infty(\Omega_T)$. They need to be discretized (homogenized) in time in order to be solved numerically. This will be the object of the next subsection. Loosely speaking, although (1.1) is associated to a fine tessellation and fine time steps, it is possible to approximate its operator on a coarse tessellation with coarse time steps.

1.3 Homogenization in space and time.

Let $M \in \mathbb{N}^*$. Let $(t_n = n\frac{T}{M})_{0\leq n\leq M}$ be a discretization of $[0,T]$. Let (φ_i) be a basis of X_h. Write Z_T the subspace of Y_T such that $w \in Z_T$ if and only if w can be written

$$w(x,t) = \sum_i c_i(t)\varphi_i(F(x,t)). \tag{1.35}$$

and the functions $t \to c_i(t)$ are constants on each intervals $(t_n, t_{n+1}]$. Write V the subspace of Y_T such that its elements ψ can be written

$$\psi(x,t) = \sum_i d_i\varphi_i(F(x,t)). \tag{1.36}$$

where the parameters d_i are constants (on $[0,T]$). For $w \in Y_T$, define $w_n \in V$ by

$$w_n(x,t) := \sum_i c_i(t_n)\varphi_i(F(x,t)). \tag{1.37}$$

Write v the solution in Z_T of the following system of implicit ordinary differential equations (such that $v(x,0) \equiv 0$): for $n \in \{0,\dots,M-1\}$ and $\psi \in V$,

$$\begin{aligned}\big(\psi(t_{n+1}), v_{n+1}(t_{n+1})\big)_{L^2(\Omega)} =& \big(\psi(t_n), v_n(t_n)\big)_{L^2(\Omega)} \\ &+ \int_{t_n}^{t_{n+1}} \Big(\big(\partial_t \psi(t), v_{n+1}(t)\big)_{L^2(\Omega)} \\ &- a\big[\psi(t), v_{n+1}(t)\big] + \big(\psi(t), g(t)\big)_{L^2(\Omega)}\Big)\, dt.\end{aligned} \tag{1.38}$$

The following theorem shows the stability of the implicit scheme (1.38).

Theorem 1.11. *Let $v \in Z_T$ be the solution of (1.38). We have*

$$\big\|v(T)\big\|_{L^2(\Omega)} + \|v\|_{L^2(0,T,H_0^1(\Omega))} \leq C\|g\|_{L^2(\Omega_T)}. \tag{1.39}$$

Remark 1.15. The constant C depends on n, Ω and $\lambda_{\min}(a)$.

The following theorem gives an error bound on the accuracy of time discretization scheme (1.38) when a does not depend on time.

Theorem 1.12. *Let $v \in Z_T$ be the solution of (1.38) and u_h be the solution of (1.31). Assume that $\partial_t a \equiv 0$. We have*

$$\begin{aligned}\big\|(u_h - v)(T)\big\|_{L^2(\Omega)} + \|u_h - v\|_{L^2(0,T,H_0^1(\Omega))} \leq& C|\Delta t| \\ &\Big(\|\partial_t g\|_{L^2(0,T,H^{-1}(\Omega))} + \big\|g(.,0)\big\|_{L^2(\Omega)}\Big).\end{aligned} \tag{1.40}$$

Remark 1.16. The constant C depends on n, Ω and $\lambda_{\min}(a)$.

The following theorem gives an error bound on the accuracy of the time discretization scheme (1.38) when a has no bounded time derivatives.

Theorem 1.13. *Assume that Ω is convex, and condition 1.1 is satisfied. Let $v \in Z_T$ be the solution of (1.38) and u_h be the solution of (1.31), we have*

$$\big\|(u_h - v)(T)\big\|_{L^2(\Omega)} + \|u_h - v\|_{L^2(0,T,H_0^1(\Omega))} \leq C\frac{|\Delta t|}{h}\|g\|_{L^2(\Omega_T)} \tag{1.41}$$

where C depends on Ω, n, δ, ϵ, $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

Remark 1.17. Observe that the accuracy of the time discretization scheme (1.38) requires that $|\Delta t| << h$ when a has no bounded time derivatives.

We refer to section 3 for numerical experiments.

1.4 Literature and further remarks.

For early works on homogenization with random mixing coefficients we refer to [59], [45], [58], [61], [73], [46], [47], [36]. Papanicolaou and Varadhan [60] have considered a two-component Markov process $(x(t), y(t))$ where $y(t)$ is rapidly varying (and is not assumed to be ergodic in dimension one) and enters in the coefficients of the stochastic process driving $x(t)$. They have studied the convergence properties of $x(t)$ as the fluctuations of $y(t)$ becomes more rapid using the martingale approach to diffusion, developed by Stroock and Varadhan [65], [64], [66], [63].

The numerical homogenization method implemented in this paper is a finite element method. The idea of using oscillating tests functions can be back tracked to the work of Murat and Tartar on homogenization and H-convergence, we refer in particular to [68] and [53]. Those papers also contain convergence proofs for the finite element method in an abstract setting for a sequence of H-converging elliptic operators (recall that the framework of H-convergence is independent from ergodicity or scale separation assumptions and are based on the compactness of any sequence of solutions of $-\operatorname{div} a_\epsilon \nabla u_\epsilon = g$ with uniformly bounded and elliptic conductivities a_ϵ, we also refer to the initial work of Spagnolo [62] for G-convergence).

The numerical implementation and practical application of oscillating test functions in numerical finite element homogenization have been called multi-scale finite element methods and have been studied by several authors [8], [28], [54], [41], [40], [34], [38], [5]. The work of Hou and Wu [41] has been a large source of inspiration in numerical applications (particularly for reservoir modeling in geophysics, we refer to [71], [48], [1] and [72] for recent developments) since it was leading to a coarse scale operator while keeping the fine scale structures of the solutions. With the method introduced by Hou and Wu, the construction of the base functions is decoupled from element to element leading to a scheme adapted to parallel computers. A proof of the convergence of the method is given in periodic settings when the size of the heterogeneities is smaller than the grid size and an oversampling technique is proposed to remove the so called cell resonance error [42] when the size of the heterogeneities is comparable to the grid size.

Allaire and Brizzi [4] have observed that multiscale finite element method with splines would have a higher accuracy and have introduced the composition rule (we also refer to [8]). In [55], it has been observed that if u is the solution of the divergence form elliptic equation

$$\begin{cases} -\operatorname{div}\big(a(x)\nabla u(x)\big) = g & \text{in} \quad \Omega \\ u = 0 \quad \text{in} \quad \partial\Omega. \end{cases} \tag{1.42}$$

and F are harmonic coordinates defined by

$$\begin{cases} \operatorname{div} a\nabla F = 0 & \text{in} \quad \Omega \\ F(x) = x & \text{on} \quad \partial\Omega. \end{cases} \tag{1.43}$$

then under the Cordes type condition $\beta_\sigma < 1$ on σ given by (1.4), one has for some $p > 2$.

$$\|u \circ F^{-1}\|_{W^{2,p}(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \tag{1.44}$$

It has been deduced from this compensation phenomenon that numerical homogenization methods based on oscillating finite elements can converge in the presence of a continuum of scales if one uses global harmonic coordinates to obtain the test functions instead of solutions of a local cell problem [55]. In dimension three and higher it has been known since the work of Fenchenko and Khruslov [33], [44] that the homogenization of divergence form elliptic operators $-\operatorname{div} a_\epsilon \nabla u_\epsilon = g$ can lead to a non local homogenized operator if the sequence of matrices a_ϵ is uniformly elliptic but with entries uniformly bounded only in $L^1(\Omega)$. From a numerical point of view this non-local effects imply that a nonlocal numerical homogenization method cannot be avoided to obtain accuracy. Hence in [55], it is shown that the accuracy of local methods depend on the aspect ratio of the triangles of the tessellation with respect to harmonic coordinates (which is not the case if one uses non local finite elements, we refer to [55] for further discussions on the apparition of non local effects in numerical homogenization). Recently Briane has shown [20] that this non-local effect is absent in dimension two in the H-convergence setting.

The phenomenon is similar here, however observe that if one has solved the initial parabolic equation at least n times and those solutions are (locally) linearly independent it is also possible to use them as new coordinates for numerical homogenization. Observe that in dimension higher than three the harmonic coordinates are not always invertible, an idea to bypass this difficulty could be either to choose the change of coordinates locally and adaptively or to enrich the coordinates by writing down the initial equations as degenerate equations in a space of higher dimension [69], these points have not been explored. For divergence form elliptic equations, recall that fast methods based on hierarchical matrices[2] are available [13, 9, 12, 10, 11] for solving (1.42) and (1.43) in $O\big(N(\ln N)^{n+3}\big)$ operations (N being the number of interior nodes of the fine mesh).

The issue of numerical homogenization partial differential equations with heterogeneous coefficients has received a great deal of attention and many methods have been proposed. A few of them are cited below.

[2]As for the fast multipole method and the hierarchical multipole method designed by L. Greengard and V. Rokhlin [39], these methods are based on the singular value decomposition of operators Green's function.

- Multi-scale finite volume methods [49].
- Heterogeneous Multi-scale Methods [27], [23].
- Wavelet based homogenization [37], [26], [24], [16], [7], [18].
- Residual free bubbles methods [19].
- Discontinuous enrichment methods [32], [31].
- Partition of Unity Methods [35].
- Energy Minimizing Multi-grid Methods [70].

Following the methods of [55], it is possible to implement a finite-volume method based on the compensation theorems given in this paper. The elements given in this paper contain the fine scale structure of F, as it has been done in [55], it is possible to approximate the initial parabolic operator by a homogenized parabolic operator associated to the coarse mesh (the test functions in this case would be piecewise linear on the coarse mesh and the approximation error associated to the homogenized operator would depend on the aspect ratio of the triangles of the coarse mesh in the metric induced by F).

Finally, in this paper a has been assumed to be bounded and uniformly elliptic. Without these assumptions the diffusion associated to homogenized operator can be anomalously slow [14], [56] or fast (super-diffusive) [57]. If a has an unbounded skew symmetric component, the homogenization of (1.1) can give rise to a degenerate operator [57].

2 Proofs

2.1 Compensation.

2.1.1 Time independent medium.

We will need the following lemmas. Let $\mathcal{A}_T$ be the bilinear form on $L^2\big(0,T;H^1_0(\Omega)\big)$ defined by

$$\mathcal{A}_T[v,w] := \int_0^T a[v,w](t)\,dt \tag{2.1}$$

where

$$a[v,u](t) := \int_\Omega {}^t\nabla v(x,t) a(x,t) \nabla u(x,t)\,dx. \tag{2.2}$$

We write $\mathcal{A}_T[u] := \mathcal{A}_T[u,u]$.

Lemma 2.1. *We have*

$$\big\|u(.,T)\big\|^2_{L^2(\Omega)} + \mathcal{A}_T[u] \leq \frac{C_{n,\Omega}}{\lambda_{\min}(a)} \|g\|^2_{L^2(\Omega_T)}. \tag{2.3}$$

Proof. Multiplying (1.1)by u and integrating with respect to time we obtain that

$$\frac{1}{2}\big\|u(.,T)\big\|^2_{L^2(\Omega)} + \mathcal{A}_T[u] = (u,g)_{L^2(\Omega_T)}. \tag{2.4}$$

Using Poincaré and Minkowksi inequalities leads us to (2.3). □

Lemma 2.2. *Assume $\partial_t a \equiv 0$. We have*

$$\big\|\partial_t u\big\|^2_{L^2(\Omega_T)} + a\big[u(.,T)\big] \leq \big\|g\big\|^2_{L^2(\Omega_T)}. \tag{2.5}$$

Proof. Multiplying (1.1) by $\partial_t u$ and integrating by parts we obtain that

$$\big\|\partial_t u(.,t)\big\|^2_{L^2(\Omega)} + a[\partial_t u, u] = (\partial_t u, g)_{L^2(\Omega)}. \tag{2.6}$$

Observing that

$$a[\partial_t u, u] = \frac{1}{2}\partial_t\big(a[u]\big) - \frac{1}{2}\int_\Omega {}^t\nabla u \partial_t a \nabla u. \tag{2.7}$$

we conclude by integration with respect to time and using Minkowski inequality. □

Lemma 2.3. *Assume $\partial_t a \equiv 0$. We have*

$$\big\|\partial_t u(.,T)\big\|^2_{L^2(\Omega)} + \mathcal{A}_T[\partial_t u] \leq \frac{C_{n,\Omega}}{\lambda_{\min}(a)}\|\partial_t g\|^2_{L^2(0,T,H^{-1}(\Omega))} + \big\|g(.,0)\big\|^2_{L^2(\Omega)}. \tag{2.8}$$

Proof. We obtain from (1.1) that

$$\partial_t^2 u = \operatorname{div}\big(a(x,t)\nabla\partial_t u(x,t)\big) + \operatorname{div}\big(\partial_t a(x,t)\nabla u(x,t)\big) + \partial_t g. \tag{2.9}$$

Multiplying (2.9) by $\partial_t u$ and integrating with respect to time we obtain that

$$\begin{aligned}\frac{1}{2}\big\|\partial_t u(.,T)\big\|^2_{L^2(\Omega)} + \mathcal{A}_T[\partial_t u] = &\int_0^T (\partial_t u, \partial_t g)_{L^2(\Omega)}dt - \int_0^T (\partial_t a)\big[\partial_t u, u\big]\,dt \\ &+ \frac{1}{2}\big\|\partial_t u(.,0)\big\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.10}$$

We conclude by the H^{-1}-duality inequality and Minkowski inequality. □

We now need a variation of Campanato's result [22] on non-divergence form elliptic operators. Let us write for a symmetric matrix M,

$$\nu_M := \frac{\operatorname{Trace}(M)}{\operatorname{Trace}({}^tMM)}. \tag{2.11}$$

We consider the following Dirichlet problem:

$$L_M v = f \tag{2.12}$$

with $L_M := \sum_{i,j=1}^n M_{ij}(x)\partial_i\partial_j$. The following theorems 2.1 and 2.2 are straightforward adaptations of theorem 1.2.1 of [51]. They are proven in [51] under the assumption that M is bounded and elliptic. It is easy to check that the conditions $\beta_M < 1$ and $\nu_M \in L^\infty(\Omega)$ are sufficient for the validity of those theorems. We refer to [55] for that adaptation.

Theorem 2.1. *Assume that $\beta_M < 1$, $\nu_M \in L^\infty(\Omega)$ and Ω is convex. Then if $f \in L^2(\Omega)$ the Dirichlet problem (2.12) has a unique solution satisfying*

$$\|v\|_{W_D^{2,2}(\Omega)} \leq \frac{C}{1-\beta_M^{\frac{1}{2}}}\|\nu_M f\|_{L^2(\Omega)}. \tag{2.13}$$

Remark 2.1. β_M is the Cordes parameter associated to M.

Theorem 2.2. *Assume that $\beta_M < 1$, $\nu_M \in L^\infty(\Omega)$ and Ω is convex. Then, there exists a real number $p_0 > 2$ depending only on n, Ω and β_M such that for each $f \in L^p(\Omega)$, $2 \leq p < p_0$ the Dirichlet problem (2.12) has a unique solution satisfying*

$$\|v\|_{W_D^{2,p}(\Omega)} \leq \frac{C_{n,\Omega,p}}{1-\beta_M^{\frac{1}{2}}}\|\nu_M f\|_{L^p(\Omega)}. \tag{2.14}$$

Let us now prove the compensation theorems. Choose

$$M := \frac{\sigma}{|\det(\nabla F)|^{\frac{1}{2}}} \circ F^{-1}. \tag{2.15}$$

It is easy to check that $\beta_\sigma < 1$ implies that F is an homeomorphism from Ω onto Ω, thus (2.15) is well defined. Moreover observe that $\beta_M = \beta_\sigma$ and

$$\|\nu_M\|^2_{L^\infty(\Omega_T)} \leq \frac{C_n}{(\lambda_{\min}(a))^{\frac{n}{2}}}\left\|(\mathrm{Trace}[\sigma])^{\frac{n}{4}-1}\right\|^2_{L^\infty(\Omega_T)}. \tag{2.16}$$

Fix $t \in [0,T]$. Choose

$$f := \frac{(\partial_t u - g)}{|\det(\nabla F)|^{\frac{1}{2}}} \circ F^{-1}. \tag{2.17}$$

Observe that by the change of variable $y = F(x)$ one obtains that if $\partial_t a \equiv 0$ (which implies that F is time independent), $\partial_t u \in L^2(\Omega)$ and $g(.,t) \in L^2(\Omega)$ that $f \in L^2(\Omega)$ and

$$\|f\|_{L^2(\Omega)} \leq \|\partial_t u\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}. \tag{2.18}$$

It follows from theorem 2.1 that there exists a unique $v \in W^{2,2}_D(\Omega)$ satisfying

$$\|v\|^2_{W^{2,2}_D(\Omega)} \leq \frac{C\|\nu_M\|^2_{L^\infty(\Omega_T)}}{(1-\beta_\sigma^{\frac{1}{2}})^2}\big(\|\partial_t u\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}\big). \tag{2.19}$$

and the following equation

$$\partial_t \hat{u}(y,t) = \sum_{i,j}\big(\sigma(F^{-1}(y,t),t)\big)_{i,j}\partial_i\partial_j v(y,t) + \hat{g}(y,t). \tag{2.20}$$

We use the notation $\hat{g} := g \circ F^{-1}$ and $\hat{u} := u \circ F^{-1}$. Using the change of variable $y = F(x)$ and using the property $\operatorname{div} a\nabla F = 0$ when $\partial_t a \equiv 0$ we obtain that (2.20) can be written

$$\partial_t u = \operatorname{div}\big(a\nabla(v \circ F)\big) + g. \tag{2.21}$$

If $\partial_t u \in L^2(\Omega)$ and $g(.,t) \in L^2(\Omega)$ we can use the uniqueness property of the solution of the divergence form elliptic Dirichlet problem

$$\operatorname{div}\big(a\nabla w\big) = \partial_t u - g. \tag{2.22}$$

to obtain that $v \circ F = u$. Thus using lemma 2.3 we have proven theorem 1.2. Moreover assume that $g \in L^2(\Omega_T)$ and $\partial_t u \in L^2(\Omega_T)$. It follows that for $t \in [0,T] - B$, $g(.,t) \in L^2(\Omega)$ and $\partial_t u(.,t) \in L^2(\Omega)$ where B is a subset of $[0,T]$ of 0-Lebesgue measure. It follows from the previous arguments that on $[0,T] - B$, $u \circ F^{-1}(.,t) \in W^{2,2}_D(\Omega)$ and satisfies

$$\|u \circ F^{-1}(.,t)\|^2_{W^{2,2}_D(\Omega)} \leq \frac{C\|\nu_M\|^2_{L^\infty(\Omega_T)}}{(1-\beta_\sigma^{\frac{1}{2}})^2}\big(\|\partial_t u(.,t)\|^2_{L^2(\Omega)} + \|g(.,t)\|^2_{L^2(\Omega)}\big). \tag{2.23}$$

Integrating (2.23) with respect to time we obtain that
$u \circ F^{-1} \in L^2(0,T,W^{2,2}_D(\Omega))$ and

$$\|u \circ F^{-1}\|^2_{L^2(0,T,W^{2,2}_D(\Omega))} \leq \frac{C\|\nu_M\|^2_{L^\infty(\Omega_T)}}{(1-\beta_\sigma^{\frac{1}{2}})^2}\big(\|\partial_t u\|^2_{L^2(\Omega_T)} + \|g\|^2_{L^2(\Omega_T)}\big). \tag{2.24}$$

Thus using lemma 2.2 we have obtained theorem 1.1.

Let us now prove theorem 1.3. Assume that there exists $q_0 > 2$ such that for $2 \leq p < q_0$, $\partial_t u \in L^p(\Omega_T)$ and $g \in L^p(\Omega_T)$. Let us now apply theorem 2.2 with $p < \min(p_0, q_0)$, M given by (2.15) and f given by (2.17). It follows that for $t \in [0,T] - B$ (where B is a subset of $[0,T]$ of 0-Lebesgue measure), $g(.,t) \in L^p(\Omega)$ and $\partial_t u(.,t) \in L^p(\Omega)$. We deduce from theorem

2.2 and the argumentation related to equation (2.22) that on $[0,T]-B$, $u\circ F^{-1}(.,t)\in W^{2,p}_D(\Omega)$ and

$$\|u\circ F^{-1}(.,t)\|^p_{W^{2,p}_D(\Omega)} \leq \frac{C_{n,p,\Omega}\|\nu_M\|^p_{L^\infty(\Omega_T)}}{(1-\beta_\sigma^{\frac{1}{2}})^p}\big(\|\partial_t u(.,t)\|^p_{L^p(\Omega)}+\|g(.,t)\|^p_{L^p(\Omega)}\big). \tag{2.25}$$

Integrating (2.25) with respect to time we obtain that
$u\circ F^{-1}\in L^p(0,T,W^{2,p}_D(\Omega))$ and

$$\|u\circ F^{-1}\|_{L^p(0,T,W^{2,p}_D(\Omega))} \leq \frac{C_{n,p,\Omega}\|\nu_M\|_{L^\infty(\Omega_T)}}{1-\beta_\sigma^{\frac{1}{2}}}\big(\|\partial_t u\|_{L^p(\Omega_T)}+\|g\|_{L^p(\Omega_T)}\big). \tag{2.26}$$

It remains to show that under the assumptions of theorem 1.3, $\partial_t u\in L^p(\Omega_T)$.

In order to bound $\big\|\partial_t u(.,t)\big\|_{L^p(\Omega)}$ we use general Sobolev inequalities (chapter 5.6 of [30]).

- If $n\geq 3$, write $p^*=2n/(n-2)$. We have for $2<p\leq p^*$,

$$\Big(\int_\Omega (\partial_t u)^p\,dx\Big)^{\frac{2}{p}} \leq C_{n,\Omega}\Big(\int_\Omega (\partial_t u)^{p^*}\,dx\Big)^{\frac{2}{p^*}} \tag{2.27}$$

  thus, using Gagliardo-Nirenberg-Sobolev inequality

$$\Big(\int_\Omega (\partial_t u)^p\,dx\Big)^{\frac{2}{p}} \leq C_{n,p,\Omega}\frac{1}{\lambda_{\min}(a)}a[\partial_t u]. \tag{2.28}$$

- If $n=2$, we write for $(x_1,x_2,x_3)\in\Omega\times(0,1)$, $v(x_1,x_2,x_3):=\partial_t u(x_1,x_2)$. Using Gagliardo-Nirenberg-Sobolev inequality in dimension three we obtain that for $2<p\leq 6$

$$\Big(\int_\Omega (\partial_t u)^p\,dx\Big)^{\frac{2}{p}} \leq C_{n,p,\Omega}\int_\Omega (\nabla\partial_t u)^2\,dx. \tag{2.29}$$

  Which leads us to (2.28).

- If $n=1$ then using Morrey's inequality we obtain that with $\gamma:=1/2$,

$$\|\partial_t u\|^2_{C^{0,\gamma}(\Omega)} \leq C_\Omega\frac{1}{\lambda_{\min}(a)}a[\partial_t u]. \tag{2.30}$$

We conclude the proof of theorem 1.3 by using lemma 2.3.

We deduce theorem 1.4 from Morrey's inequality and theorem 1.3.

Hölder continuity for $n \geq 3$ or non-convexity of Ω. In this paragraph we will not assume Ω to be convex. Let $N^{p,\lambda}(\Omega)$ ($1 < p < \infty$, $0 < \lambda < n$) be the weighted Morrey space formed by the functions $v : \Omega \to \mathbb{R}$ such that $\|v\|_{N^{p,\lambda}(\Omega)} < \infty$ with

$$\|v\|_{N^{p,\lambda}(\Omega)} := \sup_{x_0 \in \Omega} \Big(\int_\Omega |x - x_0|^{-\lambda} |v(x)|^p \Big)^{\frac{1}{p}}. \tag{2.31}$$

To obtain the Hölder continuity of $u \circ F^{-1}$ in dimension $n \geq 3$ we use corollary 4.1 of [50]. We give the result of S. Leonardi below in a form adapted to our context. Consider the Dirichlet problem (2.12). We do not assume Ω to be bounded. We write $W^{2,p,\lambda}(\Omega)$ the functions in $W^{2,p}_D(\Omega)$ such that their second order derivatives belong to $N^{p,\lambda}(\Omega)$.

Theorem 2.3. *There exist a constant $C^* = C^*(n, p, \lambda, \partial\Omega) > 0$ such that if $\beta_M < C^*$ and $f \in N^{p,\lambda}(\Omega)$ then the Dirichlet problem (2.12) has a unique solution in $W^{2,p,\lambda} \cap W^{1,p}_0(\Omega)$. Moreover, if $0 < \lambda < n < p$ then $\nabla v \in C^\alpha(\Omega)$ with $\alpha = 1 - n/p$ and*

$$\|\nabla v\|_{C^\alpha(\Omega)} \leq \frac{C}{\lambda_{\min}(M)} \|f\|_{N^{p,\lambda}(\Omega)} \tag{2.32}$$

where $C = C(n, p, \lambda, \partial\Omega)$.

The proof of theorem 1.5 is an application of theorem 2.3. We just need to observe that from Hölder inequality we have for $0 < \epsilon < 0.5$

$$\|f\|_{N^{p,\epsilon}(\Omega)} \leq C_{n,p,\Omega,\epsilon} \|f\|_{L^{p(1+\epsilon)}(\Omega)}. \tag{2.33}$$

From this point the proof is similar to the proof of theorem 1.3.

2.1.2 Medium with a continuum of time scales.

We will need theorems 1.6.2 and 1.6.3 of [51]. For the sake of completeness we will remind those theorems below in version adapted to our framework. Consider the following parabolic problem:

$$\partial_t v = \sum_{i,j=1}^{n} M_{ij}(x) \partial_i \partial_j v + f. \tag{2.34}$$

We assume M to be symmetric bounded and elliptic and $v = 0$ at $t = 0$ and on the boundary $\partial\Omega$. Write

$$\eta_M := \sup_{x \in \Omega_T} \frac{\mathrm{Trace}[{}^t M M] + 1}{\big(\mathrm{Trace}[M] + 1\big)^2}. \tag{2.35}$$

and

$$\alpha_M := \sup_{x\in\Omega_T} \frac{\mathrm{Trace}[M]+1}{\mathrm{Trace}[{}^tMM]+1}. \tag{2.36}$$

Write for $p \geq 2$

$$S_p(\Omega_T) := \Big\{ v \in L^p\big(0,T,W^{2,p}_D(\Omega)\big); \partial_t v \in L^p(\Omega_T); v(.,0) \equiv 0 \Big\} \tag{2.37}$$

and

$$\|v\|^p_{S_p(\Omega_T)} := \int_{\Omega_T} \big(\sum_{i,j} (\partial_i\partial_j v)^2 + (\partial_t v)^2\big)^{\frac{p}{2}}\, dy\, dt. \tag{2.38}$$

Theorem 2.4. *Assume Ω to be convex and that there exists $\epsilon \in (0,1)$ such that $\eta_M \leq 1/(n+\epsilon)$, then for each $f \in L^2(\Omega_T)$ the Cauchy-Dirichlet problem (2.34) admits a unique solution in $S_2(\Omega_T)$ which satisfies the bound*

$$\|v\|_{S_2(\Omega_T)} \leq \frac{\alpha_M}{1-\sqrt{1-\epsilon}} \|f\|_{L^2(\Omega_T)}. \tag{2.39}$$

Theorem 2.5. *Assume Ω to be convex and that there exists $\epsilon \in (0,1)$ such that $\eta_M \leq 1/(n+\epsilon)$, then there exists a number $p_0 > 2$ depending on Ω, n, ϵ such that for each $f \in L^p(\Omega_T)$ the Cauchy-Dirichlet problem (2.34) admits a unique solution in $S_p(\Omega_T)$ which satisfies the bound*

$$\|v\|_{S_p(\Omega_T)} \leq C_p \frac{\alpha_M}{1-\sqrt{1-\epsilon}} \|f\|_{L^p(\Omega_T)}. \tag{2.40}$$

Remark 2.2. In fact theorem 1.6.3 of [51] is written with $1 - C(p)\sqrt{1-\epsilon}$ in the denominator of (2.40) but it is easy to modify it to obtain (2.40) by lowering the value of p_0 changing the value of C_p.

Let $\delta > 0$. Let us now apply theorem 2.4 on $[0, T/\delta]$ with

$$M := \delta\sigma \circ F^{-1}(y, \delta t) \tag{2.41}$$

and

$$f := \delta(g \circ F^{-1})(y, \delta t). \tag{2.42}$$

Observe that if condition 1.1 is satisfied then F is an homeomorphism and M is well defined, bounded and elliptic. Moreover $\eta_M < \infty$ and $\alpha_M < \infty$ since

$$\mathrm{esssup}_{\Omega_{\frac{T}{\delta}}} \frac{\mathrm{Trace}[{}^tMM]+1}{\big(\mathrm{Trace}[M]+1\big)^2} = \mathrm{esssup}_{\Omega_T} \frac{\delta^2\,\mathrm{Trace}[{}^t\sigma\sigma]+1}{\big(\delta\,\mathrm{Trace}[\sigma]+1\big)^2}\,. \tag{2.43}$$

It follows that the following equation admits a unique solution in $S_2(\Omega_{\frac{T}{\delta}})$.

$$\partial_t w(y,t) = \sum_{i,j} M_{i,j}(y,t)\partial_i\partial_j w(y,t) + k(y,t) \tag{2.44}$$

with $k(y,t) = \delta\hat{g}(y,\delta t)$. And we have

$$\int_0^{\frac{T}{\delta}} \int_\Omega \big((\partial_t w)^2 + \sum_{i,j}(\partial_i\partial_j w)^2\big)\,dy\,dt \leq \frac{C}{(1-\sqrt{1-\epsilon})^2}\|f\|_{L^2(\Omega_{\frac{T}{\delta}})}. \tag{2.45}$$

Using the change of variables $t \to \delta t$ and writing

$$w(y,t) := v(y,\delta t). \tag{2.46}$$

we obtain that v satisfies the following equation on Ω_T

$$\partial_t v(y,t) = \sum_{i,j}\big(\sigma(F^{-1}(y,t),t)\big)_{i,j}\partial_i\partial_j v(y,t) + \hat{g}(y,t). \tag{2.47}$$

Using the change of variable $y = F(x)$ and observing that $\partial_t F = \operatorname{div} a\nabla F$ we obtain that $v \circ F$ satisfies

$$\partial_t(v\circ F) = \operatorname{div}\big(a\nabla(v\circ F)\big) + g. \tag{2.48}$$

It follows from the uniqueness of the solution of (2.48) that $u = v \circ F$. In resume we have obtained theorem 1.6 (we use lemma 2.4 to control the constants). The proof of 1.7 is similar and based on theorem 2.5. The proof of 1.8 follows from 1.7 and Morrey's inequality.

Let us now prove proposition 1.1. Write $x = \text{Trace}[\sigma]$ and $z = n\frac{\text{Trace}[{}^t\sigma\sigma]}{(\text{Trace}[\sigma])^2}$ (observe that $1 \leq z \leq n$). It is easy to check that condition 1.1 can be written

$$-\delta^2 x^2(\frac{\epsilon+n}{n}z - 1) + 2x\delta - (n+\epsilon-1) \geq 0. \tag{2.49}$$

Choose $\delta = n\big\|(\text{Trace}[\sigma])^{-1}\big\|_{L^\infty(\Omega_T)}$. Observing that $\delta x \geq n$ and $\delta x \leq n y_\sigma$ it is easy to conclude the proof of proposition 1.1. Similarly obtains the following lemma by straightforward computation from equation (2.49).

Lemma 2.4. *Assume that condition 1.1 is satisfied then* $\mu_\sigma < C(n,\epsilon,\delta)$

$$\big\|(\text{Trace}[\sigma])^{-1}\big\|_{L^\infty(\Omega_T)} \leq C(n,\epsilon,\delta) \tag{2.50}$$

and

$$\big\|\,\text{Trace}[\sigma]\big\|_{L^\infty(\Omega_T)} \leq C(n,\epsilon,\delta). \tag{2.51}$$

2.2 Convergence of the finite element method.

Write $\mathcal{R}_h$ the projection operator mapping $L^2\big(0,T;H^1_0(\Omega)\big)$ onto Y_T defined by: for all $v \in Y_T$

$$\mathcal{A}_T[v, u - \mathcal{R}_h u] = 0. \tag{2.52}$$

Write $\rho := u - \mathcal{R}_h u$ and $\theta := \mathcal{R}_h u - u_h$.

Lemma 2.5.

$$\frac{1}{2}\big\|(u-u_h)(T)\big\|^2_{L^2(\Omega)}+\mathcal{A}_T[u-u_h]=\int_{\Omega_T}\rho\partial_t(u-u_h)+\mathcal{A}_T[\rho,u-u_h]. \tag{2.53}$$

Proof. Subtracting (1.1) (integrated against ψ) and (1.31) we obtain that

$$\big(\psi,\partial_t(u-u_h)\big)+a[\psi,u-u_h]=0 \quad \text{for all } \psi\in V_h(t). \tag{2.54}$$

Integrating by parts with respect to time we deduce that

$$\big(\psi,(u-u_h)(.,t)\big)+a[\psi,u-u_h]=\int_{\Omega_t}\partial_t\psi(u-u_h). \tag{2.55}$$

Taking $\psi=\theta$ in (2.55) we deduce that

$$\begin{aligned}\big\|(u-u_h)(.,t)\big\|^2_{L^2(\Omega)}+\mathcal{A}_t[u-u_h]=&\int_{\Omega_t}\partial_t\theta(u-u_h)+\big(\rho,(u-u_h)(.,t)\big)\\&+\mathcal{A}_t[\rho,u-u_h].\end{aligned} \tag{2.56}$$

Observing that

$$\int_0^t\big(\partial_t\theta,u-u_h\big)+\big(\rho,(u-u_h)(.,t)\big)=\frac{1}{2}\big\|(u-u_h)(.,t)\big\|^2_{L^2(\Omega)}+\int_0^t(\rho,\partial_t(u-u_h)). \tag{2.57}$$

we deduce the lemma. □

2.2.1 Time independent medium.

Lemma 2.6.

$$\begin{aligned}\big\|(u-u_h)(T)\big\|^2_{L^2(\Omega)}+\mathcal{A}_T[u-u_h]\le 2\Big(&\|\rho\|_{L^2(\Omega_T)}\|\partial_t u-\partial_t u_h\|_{L^2(\Omega_T)}\\&+\mathcal{A}_T[\rho]\Big).\end{aligned} \tag{2.58}$$

Proof. Lemma 2.6 is a straightforward consequence of lemma 2.5 and Cauchy-Schwartz and Minkowski inequalities. □

Lemma 2.7. *We have*

$$\big\|u_h(.,T)\big\|^2_{L^2(\Omega)}+\mathcal{A}_T[u_h]\le\frac{C_{n,\Omega}}{\lambda_{\min}(a)}\|g\|^2_{L^2(\Omega_T)}. \tag{2.59}$$

Proof. Taking $\psi=u_h$ in (1.31) and integrating with respect to time we obtain that

$$\frac{1}{2}\big\|u_h(.,T)\big\|^2_{L^2(\Omega)}+\mathcal{A}_T[u_h]=(u_h,g)_{L^2(\Omega_T)}. \tag{2.60}$$

Using Poincaré and Minkowksi inequalities leads us to (2.59). □

Lemma 2.8. *Assume $\partial_t a \equiv 0$. We have*

$$\left\|\partial_t u_h\right\|^2_{L^2(\Omega_T)} + a\left[u_h(.,T)\right] \leq \left\|g\right\|^2_{L^2(\Omega_T)}. \tag{2.61}$$

Proof. The proof is similar to lemma 2.2. We need to take $\psi = \partial_t u_h$ in (1.31). □

Let $t \in [0,T]$ and $v \in H^1_0(\Omega)$, we will write $\mathcal{R}_{h,t}v(.,t)$ the solution of:

$$\int_\Omega {}^t\nabla\psi a(x,t)(\psi, v - \mathcal{R}_{h,t}v)\,dx = 0 \quad \text{for all } \psi \in V_h(t). \tag{2.62}$$

We will need the following lemma,

Lemma 2.9. *Assume the mapping $x \to F(x,t)$ to be invertible, then for $v \in H^1_0(\Omega)$ we have*

- *For $n = 1$,*

$$\left(a[v - \mathcal{R}_{h,t}v]\right)^{\frac{1}{2}} \leq C_X h \|v \circ F^{-1}(.,t)\|_{W^{2,2}_D} \|a\nabla F\|^{\frac{1}{2}}_{L^\infty(\Omega_T)}. \tag{2.63}$$

- *For $n \geq 2$,*

$$\begin{aligned}\left(a[v - \mathcal{R}_{h,t}v]\right)^{\frac{1}{2}} \leq& C_X h \|v \circ F^{-1}(.,t)\|_{W^{2,2}_D} \\ &\times C_n \mu_\sigma^{\frac{n-1}{4}} \left\|(\mathrm{Trace}[\sigma])^{-1}\right\|^{\frac{n-2}{4}}_{L^\infty(\Omega_T)}.\end{aligned} \tag{2.64}$$

Remark 2.3. Recall that μ_σ is given by equation (1.12)and it is easy to check that μ_σ is bounded by an increasing function of $(1-\beta_\sigma)^{-1}$.

Proof. Using the change of coordinates $y = F(x,t)$ we obtain that (we write $\hat{v} := v \circ F^{-1}$)

$$a[v] = Q[\hat{v}] \tag{2.65}$$

with

$$\mathcal{Q}[w] := \int_\Omega {}^t\nabla w(y,t) Q(y,t) \nabla w(y,t)\,dy \tag{2.66}$$

and

$$Q(y,t) := \frac{\sigma}{\det(\nabla F)} \circ F^{-1}. \tag{2.67}$$

Using the definition of $\mathcal{R}_{h,t}v$ we obtain that

$$\mathcal{Q}[\hat{v} - \widehat{\mathcal{R}_{h,t}v}] = \inf_{\varphi \in X_h} Q[\hat{v} - \varphi]. \tag{2.68}$$

Using property (1.26) we obtain that

$$\mathcal{Q}[\hat{v} - \widehat{\mathcal{R}_{h,t}v}] \leq \lambda_{\max}(Q) C_X^2 h^2 \|\hat{v}\|^2_{W^{2,2}_D(T)}. \tag{2.69}$$

It is easy to obtain that

- $n = 1$.

$$\lambda_{\max}(Q) \leq \|a\nabla F\|_{L^\infty(\Omega_T)}. \tag{2.70}$$

- $n \geq 2$.

$$\lambda_{\max}(Q) \leq C_n \mu_\sigma^{\frac{n-1}{2}} \left\|(\mathrm{Trace}[\sigma])^{-1}\right\|_{L^\infty(\Omega_T)}^{\frac{n}{2}-1}. \tag{2.71}$$

□

Lemma 2.10. *Assume that* $\partial_t a \equiv 0$*,* Ω *is convex,* $\beta_\sigma < 1$ *and* $(\mathrm{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)$ *then*

$$\mathcal{A}_T[\rho] \leq Ch^2 \|g\|^2_{L^2(\Omega_T)}. \tag{2.72}$$

Remark 2.4. The constant C depends on C_X, n, Ω, $\lambda_{\min}(a)$ and $\left\|(\mathrm{Trace}[\sigma])^{-1}\right\|_{L^\infty(\Omega_T)}$. If $n \geq 5$ it also depends on $\left\|\mathrm{Trace}[\sigma]\right\|_{L^\infty(\Omega_T)}$ and if $n = 1$ it also depends on $\lambda_{\max}(a)$.

Proof. The proof is a straightforward application of lemma 2.9 and theorem 1.1. Observe that in dimension one $a\nabla F = (\int_\Omega a^{-1})^{-1}$ □

Lemma 2.11. *Assume that* $\partial_t a \equiv 0$*,* Ω *is convex,* $\beta_\sigma < 1$ *and* $(\mathrm{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)$ *then*

$$\|\rho\|_{L^2(\Omega_T)} \leq Ch^2 \|g\|_{L^2(\Omega_T)}. \tag{2.73}$$

Remark 2.5. The constant C depends on C_X, n, Ω, $\lambda_{\min}(a)$ and $\left\|(\mathrm{Trace}[\sigma])^{-1}\right\|_{L^\infty(\Omega_T)}$. If $n \geq 5$ it also depends on $\left\|\mathrm{Trace}[\sigma]\right\|_{L^\infty(\Omega_T)}$ and if $n = 1$ it also depends on $\lambda_{\max}(a)$.

Proof. The proof follows from standard duality techniques (see for instance theorem 5.7.6 of [17]). We choose $v \in L^2(0, T, H^1_0(\Omega))$ to be the solution of the following linear problem: for all $w \in L^2(0, T, H^1_0(\Omega))$

$$A_T[w, v] = (w, \rho)_{L^2(\Omega_T)}. \tag{2.74}$$

Choosing $w = \rho$ in equation (2.74) we deduce that

$$\|\rho\|^2_{L^2(\Omega_T))} = \mathcal{A}_T[\rho, v - \mathcal{R}_{h,t} v]. \tag{2.75}$$

Using Cauchy Schwartz inequality we deduce that

$$\|\rho\|^2_{L^2(\Omega_T)} \leq \big(\mathcal{A}_T[\rho]\big)^{\frac{1}{2}} \big(\mathcal{A}_T[v - \mathcal{R}_{h,t} v]\big)^{\frac{1}{2}}. \tag{2.76}$$

Using theorem 1.1 we obtain that

$$\|\hat{v}\|_{L^2(0,T,W^{2,2}_D(\Omega))} \leq C\|\rho\|_{L^2(\Omega_T)}. \tag{2.77}$$

Using lemma 2.9 we obtain that

$$\big(\mathcal{A}_T[v - \mathcal{R}_{h,t} v]\big)^{\frac{1}{2}} \leq Ch \|\rho\|_{L^2(\Omega_T)}. \tag{2.78}$$

It follows that

$$\|\rho\|_{L^2(\Omega_T)} \leq Ch\big(\mathcal{A}_T[\rho]\big)^{\frac{1}{2}}. \tag{2.79}$$

We deduce the lemma by applying lemma 2.10 to bound $A_T[\rho]$. $\square$

Theorem 2.6. *Assume that $\partial_t a \equiv 0$, Ω is convex, $\beta_\sigma < 1$ and $(\mathrm{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)$ then*

$$\big\|(u-u_h)(.,T)\big\|^2_{L^2(\Omega)} + \big\|u-u_h\big\|^2_{L^2(0,T;H^1_0(\Omega))} \leq Ch^2\|g\|^2_{L^2(\Omega_T)}. \tag{2.80}$$

Remark 2.6. The constant C depends on C_X, n, Ω, $\lambda_{\min}(a)$ and $\big\|(\mathrm{Trace}[\sigma])^{-1}\big\|_{L^\infty(\Omega_T)}$. If $n \geq 5$ it also depends on $\big\|\,\mathrm{Trace}[\sigma]\big\|_{L^\infty(\Omega_T)}$ and if $n = 1$ it also depends on $\lambda_{\max}(a)$.

Proof. The proof is a straightforward application of lemmas 2.11, 2.10, 2.8, 2.6 and 2.2. $\square$

2.2.2 Medium with a continuum of time scales.

In this subsection we will assume that the finite elements are in $H^2(\Omega) \cap H^1_0(\Omega)$ and satisfy inverse inequality (1.27).

Lemma 2.12.

$$\frac{1}{2}\big\|(u-u_h)(t)\big\|^2_{L^2(\Omega)} + \mathcal{A}_t[u-u_h] = \int_{\Omega_t} \frac{\hat{\rho}}{|\det \nabla F| \circ F^{-1}} \big(\hat{g} + \sum_{i,j=1}^{n} \sigma_{i,j} \circ F^{-1}\partial_i\partial_j \hat{u}_h - \partial_t \hat{u}_h\big). \tag{2.81}$$

Proof. Consider equation (2.53). We have

$$\int_{\Omega_t} \rho\partial_t(u-u_h) = \int_{\Omega_t} \frac{\hat{\rho}}{|\det \nabla F| \circ F^{-1}} \partial_t(\hat{u}-\hat{u}_h) + \int_{\Omega_t} \rho\partial_t F(\nabla F)^{-1}\nabla(u-u_h). \tag{2.82}$$

Using equation (1.2) we obtain that

$$\int_{\Omega_t} \rho\partial_t F(\nabla F)^{-1}\nabla(u-u_h) = -\mathcal{A}_t[\rho, u-u_h] - \sum_{i,j=1}^{n}\int_{\Omega_t} \hat{\rho} Q_{i,j}\partial_i\partial_j(\hat{u}-\hat{u}_h). \tag{2.83}$$

$\square$

Lemma 2.13.

$$\Big\|\frac{\partial_t \hat{u}_h}{|\det(\nabla F)|^{\frac{1}{2}} \circ F^{-1}}\Big\|_{L^2(\Omega_T)} \leq 2\|g\|_{L^2(\Omega_T)} + C\|\hat{u}_h\|_{L^2(0,T,W^{2,2}_D(\Omega))}. \tag{2.84}$$

where the constant C depends on n, $\lambda_{\max}(a)$, $\|\,\mathrm{Trace}[\sigma]\|_{L^\infty(\Omega_T)}$, μ_σ.

Proof. Using the change of variable $y = F(x,t)$ in (1.31) we obtain that for all $\varphi \in X_h$

$$\begin{cases} (\varphi, \frac{\partial_t \hat{u}_h}{|\det(\nabla F)|\circ F^{-1}})_{L^2(\Omega)} = & \sum_{i,j=1}^n \int_\Omega (\varphi, Q_{i,j}\partial_i\partial_j \hat{u}_h)_{L^2(\Omega)} \\ & +(\varphi, \frac{\hat{g}}{|\det(\nabla F)|\circ F^{-1}})_{L^2(\Omega)} \\ \hat{u}_h(x,0) = 0. \end{cases} \tag{2.85}$$

Recall that Q is given by (2.67). We choose $\varphi = \partial_t \hat{u}$ and observe that

$$\frac{\sigma}{|\det \nabla F|^{\frac{1}{2}}} = \frac{\sigma}{|\det \sigma|^{\frac{1}{4}}} |\det a|^{\frac{1}{4}}. \tag{2.86}$$

Thus

$$\|\frac{\sigma}{|\det \nabla F|^{\frac{1}{2}}}\| \leq C\big(n, \lambda_{\max}(a), \|\operatorname{Trace}[\sigma]\|_{L^\infty(\Omega_T)}, \mu_\sigma\big). \tag{2.87}$$

We deduce the lemma by Minkowski inequality. □

Combining lemma 2.12 and lemma 2.13 we obtain the following lemma

Lemma 2.14.

$$\begin{aligned} \frac{1}{2}\|(u-u_h)(T)\|^2_{L^2(\Omega)} + \mathcal{A}_T[u-u_h] \leq & \|\rho\|_{L^2(\Omega_T)}\Big(\|g\|_{L^2(\Omega_T)} \\ & + C\|\hat{u}_h\|_{L^2(0,T,W^{2,2}_D(\Omega))}\Big). \end{aligned} \tag{2.88}$$

where the constant C depends on n, $\lambda_{\max}(a)$, $\|\operatorname{Trace}[\sigma]\|_{L^\infty(\Omega_T)}$, μ_σ.

Lemma 2.15. *Assume that Ω is convex, and condition 1.1 is satisfied then*

$$\|\rho\|_{L^2(\Omega_T)} \leq Ch^2\|g\|_{L^2(\Omega_T)}. \tag{2.89}$$

Remark 2.7. The constant C depends on C_X, n, Ω, δ and ϵ, $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

Proof. The proof is similar to the proof of lemma 2.11. As in 2.11 we choose $v \in L^2(0,T,H^1_0(\Omega))$ to be the solution of the following linear problem: for all $w \in L^2(0,T,H^1_0(\Omega))$

$$A_T[w,v] = (w,\rho)_{L^2(\Omega_T)}. \tag{2.90}$$

Choosing $w = \rho$ in equation (2.90) we deduce that

$$\|\rho\|^2_{L^2(\Omega_T))} = \mathcal{A}_T[\rho, v - \mathcal{R}_{h,t}v]. \tag{2.91}$$

Using Cauchy Schwartz inequality we deduce that

$$\|\rho\|^2_{L^2(\Omega_T)} \leq \big(\mathcal{A}_T[\rho]\big)^{\frac{1}{2}}\big(\mathcal{A}_T[v - \mathcal{R}_{h,t}v]\big)^{\frac{1}{2}}. \tag{2.92}$$

Using theorem 1.6 we obtain that

$$\|\hat{v}\|_{L^2(0,T,W_D^{2,2}(\Omega))} \leq C\|\rho\|_{L^2(\Omega_T)}. \tag{2.93}$$

Using lemma 2.9 we obtain that

$$\big(\mathcal{A}_T[v-\mathcal{R}_{h,t}v]\big)^{\frac{1}{2}} \leq Ch\|\rho\|_{L^2(\Omega_T)}. \tag{2.94}$$

It follows that

$$\|\rho\|_{L^2(\Omega)} \leq Ch\big(\mathcal{A}_T[\rho]\big)^{\frac{1}{2}}. \tag{2.95}$$

We deduce the lemma by applying lemma 2.9 and theorem 1.6 to bound $A_T[\rho]$. □

Lemma 2.16. *Assume that* Ω *is convex, and that* $\mathrm{Trace}[\sigma] \in L^\infty(\Omega_T)$.

$$\|\hat{u}_h\|_{L^2(0,T,W_D^{2,2}(\Omega))} \leq \frac{C}{h}\|g\|_{L^2(\Omega_T)}. \tag{2.96}$$

Remark 2.8. The constant C depends on C_X, n, Ω, $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$ and $\|\mathrm{Trace}[\sigma]\|_{L^\infty(\Omega_T)}$.

Proof. Using the inverse inequality (1.27) of the finite elements we obtain that

$$\|\hat{u}_h\|_{L^2(0,T,W_D^{2,2}(\Omega))} \leq \frac{C_X}{h}\|\nabla\hat{u}_h\|_{L^2(0,T,W_D^{2,2}(\Omega))}. \tag{2.97}$$

Using the change of variables $y = F(x)$ we obtain that

$$\|\nabla\hat{u}_h\|^2_{L^2(0,T,W_D^{2,2}(\Omega))} \leq C\mathcal{A}_T[u_h] \tag{2.98}$$

where C depends on n, $\lambda_{\min}(a)$ and $\|\mathrm{Trace}[\sigma]\|_{L^\infty(\Omega_T)}$. We deduce the lemma by using lemma 2.7. □

Theorem 2.7. *Assume that* Ω *is convex, and condition 1.1 is satisfied then*

$$\frac{1}{2}\big\|(u-u_h)(T)\big\|^2_{L^2(\Omega)} + \mathcal{A}_T[u-u_h] \leq Ch\|g\|^2_{L^2(\Omega_T)}. \tag{2.99}$$

Remark 2.9. The constant C depends on C_X, n, Ω, δ and ϵ, $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

Proof. The proof is a straightforward application of lemma 2.14, lemma 2.15 and lemma 2.16. □

2.2.3 Homogenization and discretization in time.

We use the notation of subsection 1.3. First let us observe that the numerical scheme associated to (1.38) is stable. Indeed choosing $\psi = v_{n+1}$ one gets

$$\begin{aligned} \big|v_{n+1}(t_{n+1})\big|^2_{L^2(\Omega)} =&\big(v_{n+1}(t_n), v_n(t_n)\big)_{L^2(\Omega)} \\ &+\frac{1}{2}\big(\big|v_{n+1}(t_{n+1})\big|^2_{L^2(\Omega)} - \big|v_{n+1}(t_n)\big|^2_{L^2(\Omega)}\big) \\ &- \int_{t_n}^{t_{n+1}} \Big(a\big[v_{n+1}(t)\big] + \big(v_{n+1}(t), g(t)\big)_{L^2(\Omega)}\Big)\, dt. \end{aligned} \tag{2.100}$$

It follows by Cauchy-Schwartz inequality that

$$\begin{aligned} \frac{1}{2}\big|v_{n+1}(t_{n+1})\big|^2_{L^2(\Omega)} \leq& \frac{1}{2}\big|v_n(t_n)\big|^2_{L^2(\Omega)} \\ &- \int_{t_n}^{t_{n+1}} \Big(a\big[v_{n+1}(t)\big] + \big(v_{n+1}(t), g(t)\big)_{L^2(\Omega)}\Big)\, dt. \end{aligned} \tag{2.101}$$

Hence using Poincaré and Minkowski inequalities one obtains that

$$\begin{aligned} \big|v_{n+1}(t_{n+1})\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a\big[v_{n+1}(t)\big]\, dt \leq& \big|v_n(t_n)\big|^2_{L^2(\Omega)} \\ &+ \frac{C_{n,\Omega}}{\lambda_{\min}(a)} \int_{t_n}^{t_{n+1}} \big|g(t)\big|^2_{L^2(\Omega)}\, dt. \end{aligned} \tag{2.102}$$

which implies theorem 1.11 and the stability of the scheme. Integrating (1.31) with respect to time we obtain that for $\psi \in V$,

$$\begin{aligned} \big(\psi(t_{n+1}), u_h(t_{n+1})\big)_{L^2(\Omega)} =& \big(\psi(t_n), u_h(t_n)\big)_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} \Big(\big(\partial_t\psi(t), u_h(t)\big)_{L^2(\Omega)} \\ &- a\big[\psi(t), u_h(t)\big] + \big(\psi(t), g(t)\big)_{L^2(\Omega)}\Big)\, dt. \end{aligned} \tag{2.103}$$

Let us write (u^i) the coordinates of u_h associated to the basis $(\varphi_i \circ F)$, i.e.

$$u_h(x,t) := \sum_i u^i(t)\varphi_i(F(x,t)). \tag{2.104}$$

Let us define

$$u_n(x,t) := \sum_i u^i(t_n)\varphi_i(F(x,t)). \tag{2.105}$$

Subtracting (2.103) and (1.38) we obtain that for $\psi \in Z_T$,

$$\begin{aligned} \big(\psi(t_{n+1}), (u_{n+1} - v_{n+1})(t_{n+1})\big)_{L^2(\Omega)} =& \big(\psi(t_n), (u_n - v_n)(t_n)\big)_{L^2(\Omega)} \\ &+ \int_{t_n}^{t_{n+1}} \Big(\big(\partial_t\psi(t), (u_h - v_{n+1})(t)\big)_{L^2(\Omega)} \\ &- a\big[\psi(t), (u_h - v_{n+1})(t)\big]\Big)\, dt. \end{aligned} \tag{2.106}$$

Choosing $\psi = u_{n+1} - v_{n+1}$ we deduce using Cauchy-Schwartz inequality that

$$\begin{aligned}
&\frac{1}{2}\big|(u_{n+1}-v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt \leq \\
&\quad \frac{1}{2}\big|(u_n-v_n)(t_n)\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} \Big(\big(\partial_t(u_{n+1}-v_{n+1})(t),(u_h-u_{n+1})(t)\big)_{L^2(\Omega)} \\
&\quad - a\big[(u_{n+1}-v_{n+1})(t),(u_h-u_{n+1})(t)\big]\Big)\,dt.
\end{aligned} \tag{2.107}$$

Time independent medium. Observe that if the medium is time independent then (2.107) can be written

$$\begin{aligned}
&\frac{1}{2}\big|(u_{n+1}-v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt \leq \\
&\quad \frac{1}{2}\big|(u_n-v_n)(t_n)\big|^2_{L^2(\Omega)} - \int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t),(u_h-u_{n+1})(t)\big]\,dt
\end{aligned} \tag{2.108}$$

which leads us to

$$\begin{aligned}
&\frac{1}{2}\big|(u_{n+1}-v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt \leq \\
&\quad \frac{1}{2}\big|(u_n-v_n)(t_n)\big|^2_{L^2(\Omega)} \\
&\quad + \int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} 1(t<s)\, a\big[(u_{n+1}-v_{n+1})(t),\partial_s u_h(s)\big]\,ds\,dt.
\end{aligned} \tag{2.109}$$

Write $\Delta t := t_{n+1} - t_n$. Using Minkowski inequality we obtain that

$$\begin{aligned}
a\big[(u_{n+1}-v_{n+1})(t),\partial_s u_h(s)\big] \leq &\frac{1}{2\Delta t} a\big[(u_{n+1}-v_{n+1})(t)\big] \\
&+ \frac{1}{2}\Delta t a\big[\partial_s u_h(s)\big].
\end{aligned} \tag{2.110}$$

It follows from (2.109) that

$$\begin{aligned}
&\big|(u_{n+1}-v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt \leq \\
&\big|(u_n-v_n)(t_n)\big|^2_{L^2(\Omega)} + |\Delta t|^2 \int_{t_n}^{t_{n+1}} a\big[\partial_s u_h(s)\big]\,ds.
\end{aligned} \tag{2.111}$$

Observing that

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt \geq &0.5 \int_{t_n}^{t_{n+1}} a\big[(u_h-v_{n+1})(t)\big]\,dt \\
&- \int_{t_n}^{t_{n+1}} a\big[(u_h-u_{n+1})(t)\big]\,dt
\end{aligned} \tag{2.112}$$

and

$$\int_{t_n}^{t_{n+1}} a\big[(u_h - u_{n+1})(t)\big]\,dt \leq |\Delta t|^2 \int_{t_n}^{t_{n+1}} a\big[\partial_s u_h(s)\big]\,ds \tag{2.113}$$

we obtain that

$$\begin{aligned}&\big|(u_{n+1} - v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + 0.5\int_{t_n}^{t_{n+1}} a\big[(u_h - v_{n+1})(t)\big]\,dt \leq \\ &\big|(u_n - v_n)(t_n)\big|^2_{L^2(\Omega)} + \frac{3}{2}|\Delta t|^2 \int_{t_n}^{t_{n+1}} a\big[\partial_s u_h(s)\big]\,ds.\end{aligned} \tag{2.114}$$

In conclusion we have obtained the following following lemma

Lemma 2.17. *Let $v \in Z_T$ be the solution of (1.38). We have*

$$\big\|(u_h - v)(T)\big\|^2_{L^2(\Omega)} + \int_0^T a\big[(u_h - v)(t)\big]\,dt \leq 3|\Delta t|^2 \int_0^T a\big[\partial_s u_h(s)\big]\,ds. \tag{2.115}$$

Combining lemma 2.3 with lemma 2.17 we obtain the following theorem:

Theorem 2.8. *Let $v \in Z_T$ be the solution of (1.38). We have*

$$\begin{aligned}\big\|(u_h - v)(T)\big\|^2_{L^2(\Omega)} + \int_0^T a\big[(u_h - v)(t)\big]\,dt \leq 3|\Delta t|^2 \\ \Big(\frac{4}{\lambda_{\min}(a)}\|\partial_t g\|^2_{L^2(0,T,H^{-1}(\Omega))} + \big\|g(.,0)\big\|^2_{L^2(\Omega)}\Big).\end{aligned} \tag{2.116}$$

Time dependent medium. Observe that

$$\partial_t(u_{n+1} - v_{n+1}) = \partial_t F(\nabla F)^{-1}\nabla(u_{n+1} - v_{n+1}).$$

It follows after writing $\partial_t F = \operatorname{div} a\nabla F$, integration by parts and using the change of variables $y = F(x,t)$ in (2.107) that

$$\begin{aligned}&\frac{1}{2}\big|(u_{n+1} - v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a\big[(u_{n+1} - v_{n+1})(t)\big]\,dt \leq \\ &\frac{1}{2}\big|(u_n - v_n)(t_n)\big|^2_{L^2(\Omega)} - 2\int_{t_n}^{t_{n+1}} a\big[(u_{n+1} - v_{n+1})(t), (u_h - u_{n+1})(t)\big]\,dt \\ &- \sum_{i,j}\int_{t_n}^{t_{n+1}}\int_\Omega (\hat{u}_h - \hat{u}_{n+1})Q_{i,j}\partial_i\partial_j(\hat{u}_{n+1} - \hat{v}_{n+1})\,dt\,dy.\end{aligned} \tag{2.117}$$

Hence using Minkowski inequality we obtain that

$$\begin{aligned}
&\big|(u_{n+1}-v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt \leq \\
&\big|(u_n-v_n)(t_n)\big|^2_{L^2(\Omega)} + 4\int_{t_n}^{t_{n+1}} a\big[(u_h-u_{n+1})(t)\big]\,dt \\
&-2\sum_{i,j}\int_{t_n}^{t_{n+1}}\int_\Omega (\hat{u}_h-\hat{u}_{n+1})Q_{i,j}\partial_i\partial_j(\hat{u}_{n+1}-\hat{v}_{n+1})\,dt\,dy.
\end{aligned} \tag{2.118}$$

Using Minkowski inequality we obtain that

$$\begin{aligned}
\Big|\sum_{i,j}\int_{t_n}^{t_{n+1}}\int_\Omega (\hat{u}_h-\hat{u}_{n+1})Q_{i,j}\partial_i\partial_j(\hat{u}_{n+1}-\hat{v}_{n+1})\,dt\,dy\Big| \leq \\
C_A n^2\int_{t_n}^{t_{n+1}}\int_\Omega |\hat{u}_h-\hat{u}_{n+1}|^2\,dt\,dy \\
+\frac{\lambda_{\max}(Q)}{C_A}\int_{t_n}^{t_{n+1}}\sum_{i,j}\int_\Omega |\partial_i\partial_j(\hat{u}_{n+1}-\hat{v}_{n+1})|^2\,dt\,dy.
\end{aligned} \tag{2.119}$$

Using the inverse inequality (1.27) and the change of variable $y=F(x)$ we obtain that

$$\begin{aligned}
\int_{t_n}^{t_{n+1}}\sum_{i,j}\int_\Omega |\partial_i\partial_j(\hat{u}_{n+1}-\hat{v}_{n+1})|^2\,dt\,dy \leq \frac{C_X}{h^2\lambda_{\min}(Q)} \\
\int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt.
\end{aligned} \tag{2.120}$$

In resume, choosing $C_A=\frac{4C_X\lambda_{\max}(Q)}{h^2\lambda_{\min}(Q)}$ we have obtained that

$$\begin{aligned}
&\big|(u_{n+1}-v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + 0.5\int_{t_n}^{t_{n+1}} a\big[(u_{n+1}-v_{n+1})(t)\big]\,dt \leq \\
&\big|(u_n-v_n)(t_n)\big|^2_{L^2(\Omega)} + 8\int_{t_n}^{t_{n+1}} a\big[(u_h-u_{n+1})(t)\big]\,dt \\
&+\frac{8C_X\lambda_{\max}(Q)}{h^2\lambda_{\min}(Q)}n^2\int_{t_n}^{t_{n+1}}\int_\Omega |\hat{u}_h-\hat{u}_{n+1}|^2\,dt\,dy.
\end{aligned} \tag{2.121}$$

And a computation similar to the one leading to (2.110) gives us

$$\begin{aligned}
&\big|(u_{n+1}-v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \frac{1}{4}\int_{t_n}^{t_{n+1}} a\big[(u_h-v_{n+1})(t)\big]\,dt \leq \\
&\big|(u_n-v_n)(t_n)\big|^2_{L^2(\Omega)} + 9\int_{t_n}^{t_{n+1}} a\big[(u_h-u_{n+1})(t)\big]\,dt \\
&+\frac{8C_X\lambda_{\max}(Q)}{h^2\lambda_{\min}(Q)}n^2\int_{t_n}^{t_{n+1}}\int_\Omega |\hat{u}_h-\hat{u}_{n+1}|^2\,dt\,dy.
\end{aligned} \tag{2.122}$$

Moreover using the change of variables $F(x) = y$ and the inverse inequality (1.28) we obtain that

$$\int_{t_n}^{t_{n+1}} a\big[(u_h - u_{n+1})(t)\big]\, dt \le \frac{C_X \lambda_{\max}(Q)}{h^2} \int_{t_n}^{t_{n+1}} \int_\Omega |\hat{u}_h - \hat{u}_{n+1}|^2 \, dt\, dy. \tag{2.123}$$

Let us also observe that

$$\int_{t_n}^{t_{n+1}} \int_\Omega |\hat{u}_h - \hat{u}_{n+1}|^2 \, dt\, dy \le |\Delta t|^2 \int_{t_n}^{t_{n+1}} \int_\Omega |\partial_t \hat{u}_h|^2 \, dt\, dy. \tag{2.124}$$

It follows that

$$\begin{aligned} &\big|(u_{n+1} - v_{n+1})(t_{n+1})\big|^2_{L^2(\Omega)} + \frac{1}{4} \int_{t_n}^{t_{n+1}} a\big[(u_h - v_{n+1})(t)\big]\, dt \le \\ &\big|(u_n - v_n)(t_n)\big|^2_{L^2(\Omega)} + C_B \frac{|\Delta t|^2}{h^2} \int_{t_n}^{t_{n+1}} \int_\Omega |\partial_t \hat{u}_h|^2 \, dt\, dy. \end{aligned} \tag{2.125}$$

with

$$C_B = C_n C_X \lambda_{\max}(Q)(1 + \frac{1}{\lambda_{\min}(Q)}). \tag{2.126}$$

We deduce that

$$\big\|(u_h - v)(T)\big\|^2_{L^2(\Omega)} + \frac{1}{4} \int_0^T a\big[(u_h - v)(t)\big]\, dt \le C_B \frac{|\Delta t|^2}{h^2} \int_0^T \int_\Omega |\partial_t \hat{u}_h|^2 \, dt\, dy. \tag{2.127}$$

Using lemma 2.4 to control C_B and combining (2.127) with theorem 1.6 we obtain the following theorem.

Theorem 2.9. *Assume that Ω is convex, and condition 1.1 is satisfied. Let $v \in Z_T$ be the solution of (1.38), we have*

$$\big\|(u_h - v)(T)\big\|^2_{L^2(\Omega)} + \frac{1}{4} \int_0^T a\big[(u_h - v)(t)\big]\, dt \le C \frac{|\Delta t|^2}{h^2} \|g\|^2_{L^2(\Omega_T)}. \tag{2.128}$$

where C depends on Ω, n, δ, ϵ, $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

3 Numerical Experiments

The purpose of this section is to give several illustrations of the implementation of this method. The domain is the unit square in dimension two. Equation (1.1) is solved on a fine tessellation characterized by 16129 interior nodes (degree of freedoms).

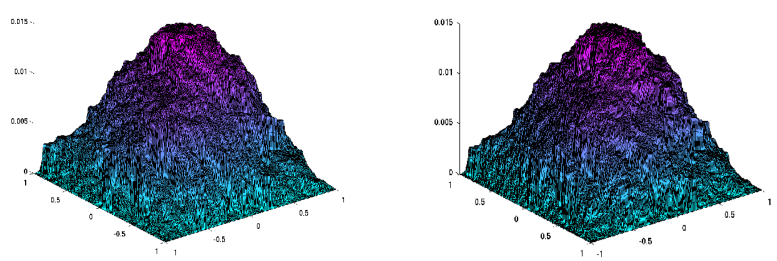
Three different coarse tessellations are considered, one with 9 degrees of freedoms (noted *dof* in the tables), one with 49 and the last one with 225.

The parabolic operator associated to equation (1.1) has been homogenized onto these coarse meshes using the method the method presented in this paper. We have chosen splines to span the space X_h introduced in subsection 1.2.

3.1 Time independent examples.

Example 1. Time independent site percolation.

In this example we consider the site percolating medium associated to figure 1. The fine mesh is characterized by 16641 nodes. (1.1) has been homogenized to three different coarse meshes with 9, 49 and 225 interior nodes using the method described here and splines for the space X_h. (1.1) has been solved with the fine mesh operator and the coarse mesh operators with $g = 1$ and $g = \sin(2.4x - 1.8y + 2\pi t)$. The fine mesh and coarse mesh errors are given in tables 1, 2, 3, 4. Figure 7 shows u computed on 16641 interior nodes and u_h computed on 9 interior nodes in the case $g = 1$ at time 1.



(a) u. **(b)** u_h.

Figure 7: u computed on 16641 interior nodes and u_h computed on 9 interior nodes.

Table 1: Coarse Mesh Error. Time Independent Site Percolation with $g = 1$.

dof	L^1	L^∞	L^2	H^1
9	0.0142	0.0389	0.0168	0.0366
49	0.0077	0.0450	0.0101	0.0482
225	0.0035	0.0228	0.0060	0.0293

Table 2: Fine Mesh Error. Time Independent Site Percolation with $g = 1$.

dof	L^1	L^∞	L^2	H^1
9	0.0196	0.0843	0.0251	0.1193
49	0.0136	0.0698	0.0184	0.1028
225	0.0040	0.0243	0.0070	0.0485

Table 3: Coarse Mesh Error. Time Independent Percolation Case with $g = \sin(2.4x - 1.8y + 2\pi t)$.

dof	L^1	L^∞	L^2	H^1
9	0.0236	0.0569	0.0262	0.0477
49	0.0181	0.0571	0.0215	0.0558
225	0.0119	0.0774	0.0167	0.0939

Table 4: Fine Mesh Error. Time Independent Percolation with $g = \sin(2.4x - 1.8y + 2\pi t)$.

dof	L^1	L^∞	L^2	H^1
9	0.0424	0.1099	0.0512	0.1712
49	0.0277	0.0985	0.0348	0.1451
225	0.0174	0.0886	0.0242	0.1192

Table 5: Coarse Mesh Error, high conductivity channel.

dof	L^1	L^∞	L^2	H^1
9	0.0159	0.0496	0.0207	0.0477
49	0.0067	0.0389	0.0102	0.0345
225	0.0035	0.0228	0.0060	0.0293

Table 6: Fine Mesh Error, high conductivity channel.

dof	L^1	L^∞	L^2	H^1
9	0.0178	0.0564	0.0257	0.0947
49	0.0079	0.0388	0.0129	0.0660
225	0.0040	0.0243	0.0070	0.0485

Example 2. Time independent high conductivity channel.

In this example a is random and characterized by a fine and long ranged high conductivity channel. We choose $a(x) = 100$, if x is in the channel, and $a(x) = O(1)$ and random, if x is not in the channel. The media is illustrated in figure 8

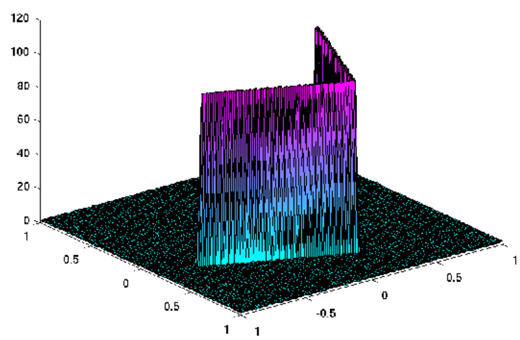


Figure 8: High Conductivity Channel superposed on a random medium.

Tables 5 and 6 give the coarse and fine meshes errors.

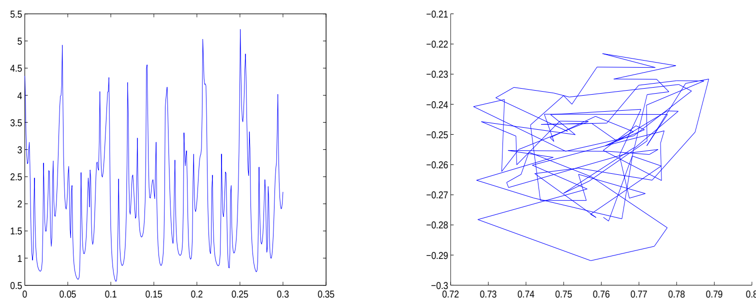
3.2 Time dependent examples.

In the following examples we consider media characterized by a continuum of time scales. In the following examples the ODE obtained on the coarse mesh from the homogenization of (1.1) have also been homogenized in time according to the method described in subsection 1.3.

Example 3. Time Dependent Multiscale trigonometric.

In this example a is given by equation (1.25). Although the number fine time steps to solve (1.1) is 2663, only 134 coarse time steps have been used to solve the homogenized equation. Hence if one also takes into account homogenization in space, the compression factor is of the order of 35000 for the coarse mesh with 9 interior nodes.

Figure 9 shows the curves of $t \to a(x_0, t)$ and $t \to F(x_0, t)$ for a given $x_0 \in \Omega$.



(a) $t \to a(x_0, t)$**.**

(b) ***Top view of*** $t \to F(x_0, t)$**.**

Figure 9: Multiscale time dependent trigonometric medium.

The coarse and fine mesh relative L^1, L^2, L^∞, and H^1 errors with respect to time have been plotted in figures 10, 11, 12 and 13. The initial increase of the relative error has its origin in the initial value $u \equiv 0$ at time 0.

The coarse and fine meshes errors are given in tables 7 and 8 for $g = 1$ at $t = 0.1$, those errors are given in tables 9 and 10 for $g = \sin(2.4x - 1.8y + 2\pi t)$ at $t = 0.1$

Table 7: Coarse Mesh Error. Multiscale trigonometric time dependent Medium. $g = 1$.

dof	L^1	L^∞	L^2	H^1
9	0.0018	0.0045	0.0019	0.0039
49	0.0012	0.0054	0.0015	0.0060

Table 8: Fine Mesh Error. Multiscale trigonometric time dependent Medium. $g = 1$.

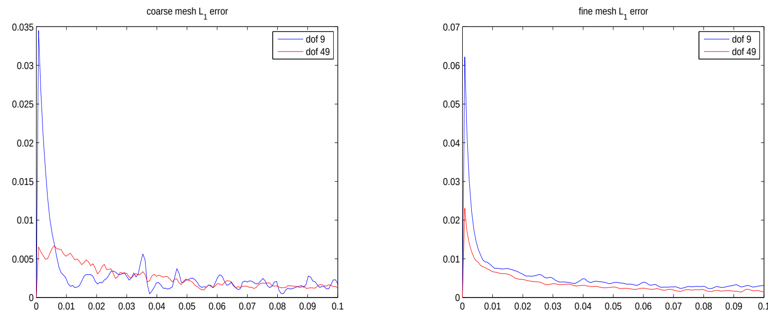
dof	L^1	L^∞	L^2	H^1
9	0.0031	0.0096	0.0034	0.0242
49	0.0014	0.0059	0.0016	0.0166

Table 9: Coarse mesh error. Multiscale trigonometric time dependent Medium. $g = \sin(2.4x - 1.8y + 2\pi t)$.

dof	L^1	L^∞	L^2	H^1
9	0.0043	0.0087	0.0044	0.0085
49	0.0033	0.0079	0.0035	0.0084

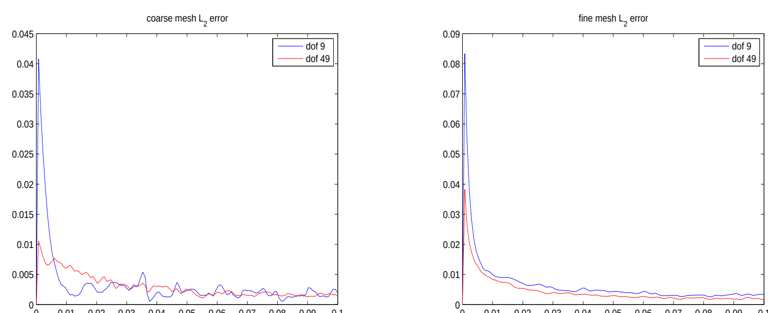
Table 10: Fine mesh error. Multiscale trigonometric time dependent medium. $g = \sin(2.4x - 1.8y + 2\pi t)$.

dof	L^1	L^∞	L^2	H^1
9	0.0082	0.0199	0.0087	0.0379
49	0.0038	0.0104	0.0040	0.0244



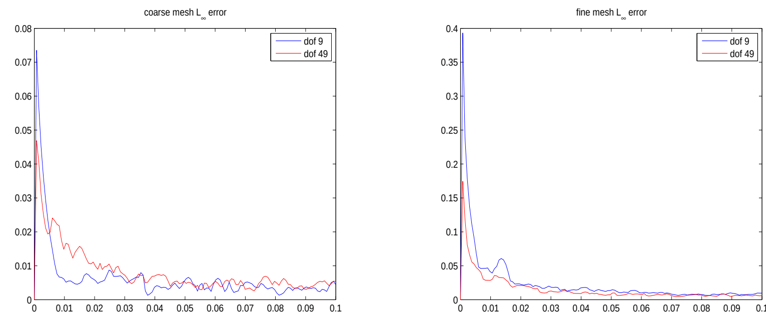
(a) ***Coarse mesh*** L^1 ***error.*** (b) ***Fine mesh*** L^1 ***error.***

Figure 10: L^1 error. Multiscale Trigonometric time dependent Medium.



(a) ***Coarse mesh*** L^2 ***error.*** (b) ***Fine Mesh*** L^2 ***error.***

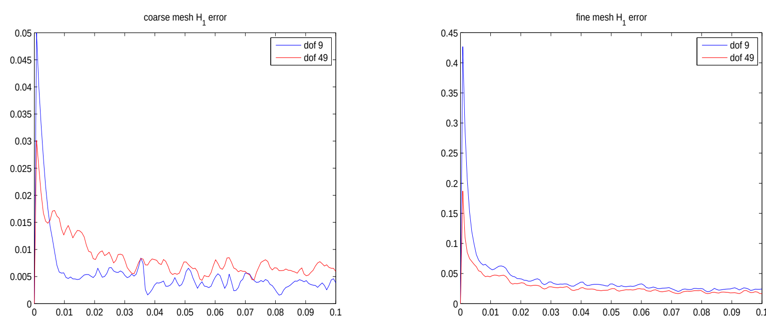
Figure 11: L^2 error. Multiscale Trigonometric time dependent Medium.



(a) ***Coarse mesh*** L_∞ ***error.***

(b) ***Fine mesh*** L_∞ ***error.***

Figure 12: L^∞ error. Multiscale Trigonometric time dependent Medium.



(a) ***Coarse Mesh*** H^1 ***error.***

(b) ***Fine Mesh*** H^1 ***error.***

Figure 13: H^1 error. Multiscale Trigonometric time dependent Medium.

Example 4. Time Dependent Random Fourier Modes.

In this example $a(x, y, t) = e^{h(x,y,t)}$ where h is given by the following equation

$$h(x, y, t) = \sum_{|k| \le R} (a_k \sin(2\pi k.x') + b_k \cos(2\pi k.x'))$$

where $R = 4$, $x' = x + \sqrt{2}t$, $y' = y - \sqrt{2}t$, a_k and b_k are independent identically distributed random variables on $[-0.2, 0.2]$. In this example, one can compute that $\frac{\lambda_{\max}(a)}{\lambda_{\min}(a)} = 95.7$.

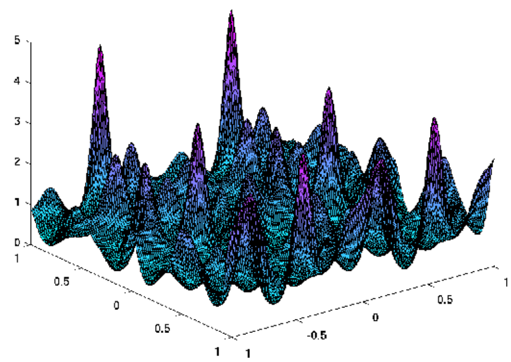
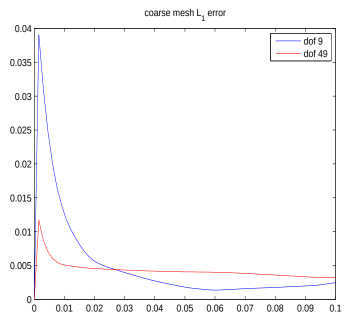


Figure 14: Random Fourier Mode

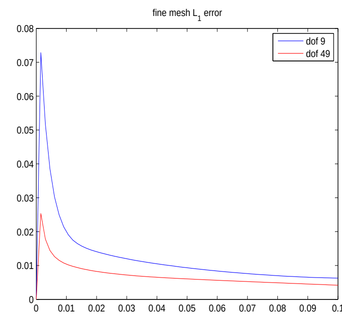
Figure 14 is a plot of a at time 0. Although the number fine time steps to solve (1.1) is 1332 only 67 coarse time steps have been used to solve the homogenized equation. Hence if one also takes into account homogenization in space, the compression factor is of the order of 35000 for the coarse mesh with 9 interior nodes.

The coarse and fine mesh relative L^1, L^2, L^∞, and H^1 errors with respect to time (up to time $t = 0.1$) have been plotted in figures 15, 16, 17 and 18. Those errors are also given up to $t = 1$ in figures 19, 20, 21, 22.

The coarse and fine meshes errors are given in tables 11 and 12 for $g = 1$ at $t = 0.1$.



(a) ***coarse mesh L^1 error.***



(b) ***fine mesh L^1 error.***

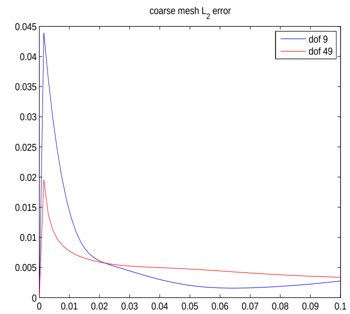
Figure 15: L^1 error. Random Fourier modes up to $t = .1$.

Table 11: Coarse mesh error at $t = 0.1$. Time dependent random Fourier modes.

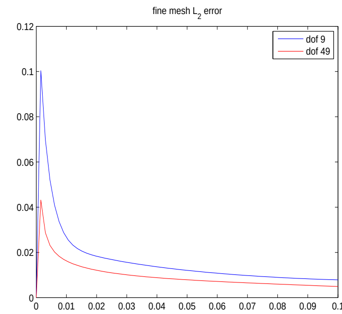
dof	L^1	L^∞	L^2	H^1
9	0.0025	0.0028	0.0064	0.0052
49	0.0032	0.0098	0.0034	0.0100

Table 12: Fine mesh error at $t = 0.1$. Time dependent random Fourier modes.

dof	L^1	L^∞	L^2	H^1
9	0.0063	0.0344	0.0079	0.0481
49	0.0042	0.0207	0.0049	0.0337

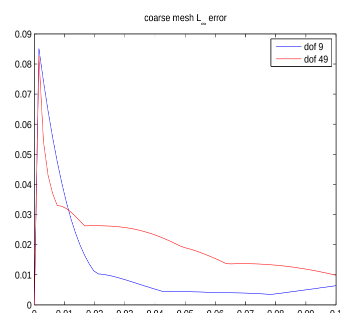


(a) ***coarse mesh*** L^2 ***error.***

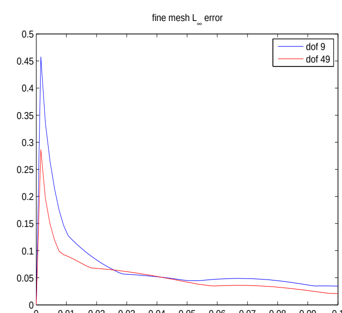


(b) ***Fine Mesh*** L^2 ***error.***

Figure 16: L^2 error. Random Fourier modes up to $t = .1$..

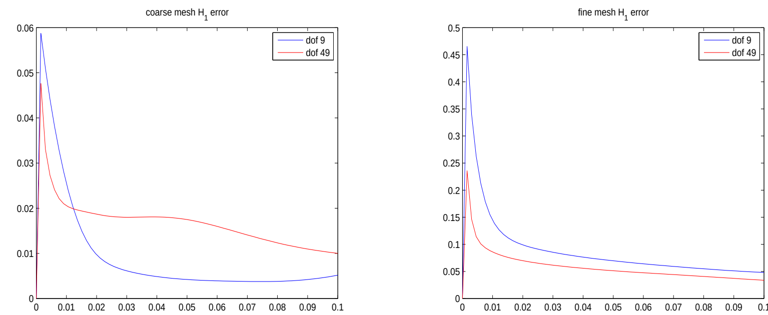


(a) ***coarse mesh*** L_∞ ***error.***



(b) ***fine mesh*** L_∞ ***error.***

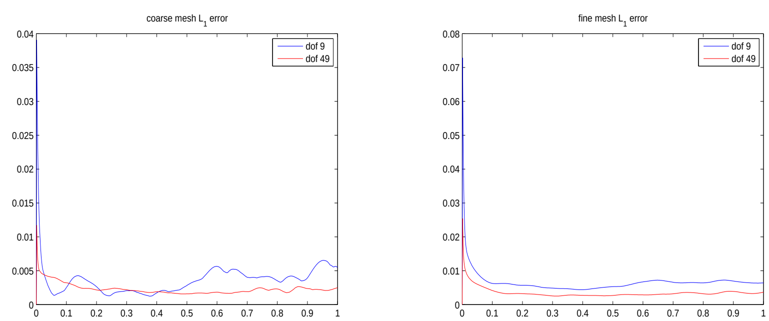
Figure 17: L^∞ error. Random Fourier modes up to $t = .1$..



(a) ***coarse mesh*** H^1 ***error.***

(b) ***fine mesh*** H^1 ***error.***

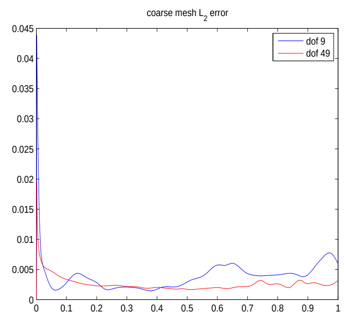
Figure 18: H^1 error. Random Fourier modes up to $t = .1$..



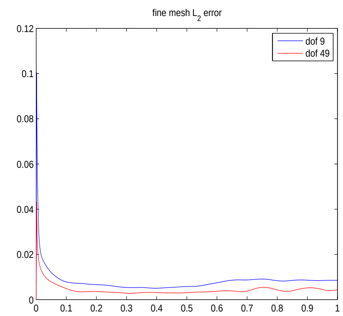
(a) ***coarse mesh*** L^1 ***error.***

(b) ***fine mesh*** L^1 ***error.***

Figure 19: L^1 error. Random Fourier modes up to $t = 1$.

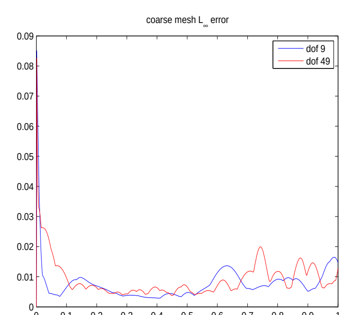


(a) ***coarse mesh*** L^2 ***error.***

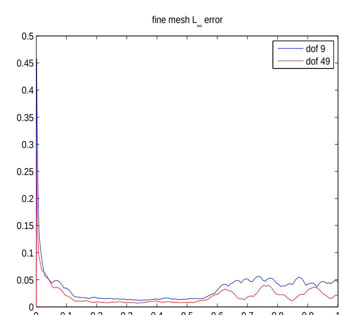


(b) ***Fine Mesh*** L^2 ***error.***

Figure 20: L^2 error. Random Fourier modes up to $t = 1$.

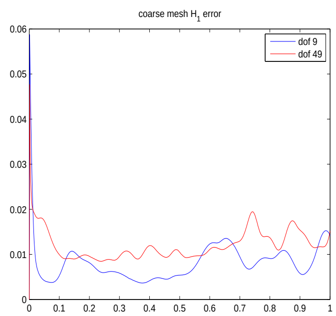


(a) ***coarse mesh*** L_∞ ***error.***

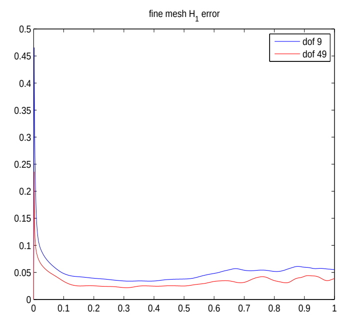


(b) ***fine mesh*** L_∞ ***error.***

Figure 21: L^∞ error. Random Fourier modes up to $t = 1$.



(a) ***coarse Mesh*** H^1 ***error.***



(b) ***fine Mesh*** H^1 ***error.***

Figure 22: H^1 error. Random Fourier modes up to $t = 1$.

Table 13: Coarse Mesh Error for the time dependent random fractal medium with spline element

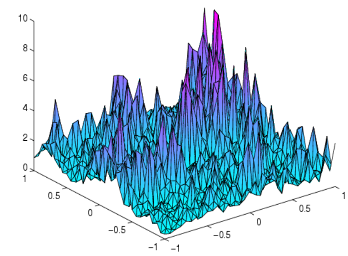
dof	L^1	L^∞	L^2	H^1
9	0.0046	0.0074	0.0052	0.0065
49	0.0036	0.0046	0.0036	0.0059

Table 14: Fine Mesh Error for the time dependent random fractal medium with spline element

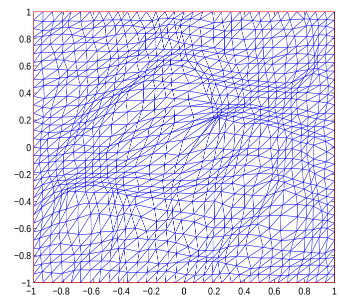
dof	L^1	L^∞	L^2	H^1
9	0.0039	0.0082	0.0043	0.0222
49	0.0033	0.0054	0.0034	0.0168

Example 5. Time Dependent Random Fractal

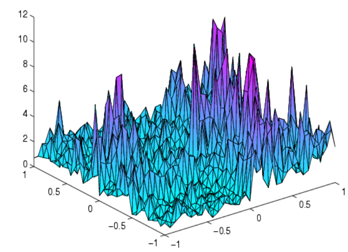
In this example, a is given by a product of discontinuous functions oscillating randomly at different space and time scales. Namely $a(x,t) := a_1(x,t)a_2(x,t)\cdots a_n(x,t)$ with $n=6$ and $a_i(x,t) = c^i_{pq}(t)$ for $x \in [\frac{p}{2^i}, \frac{p+1}{2^i}) \times [\frac{q}{2^i}, \frac{q+1}{2^i})$. The coefficients $c^i_{pq}(t)$ are chosen at random with uniform law in $[\frac{1}{\gamma}, \gamma]$ with $\gamma = 0.7$ and independently in subdivision in space and in time, thus they are assumed to be constant in each time interval $0.1 \times [\frac{k}{4^i}, \frac{k+1}{4^i})$. In this example we have $\frac{\lambda_{\max}(a)}{\lambda_{\min}(a)} = 160.3295$. Although the number of fine time steps to solve (1.1) is 3482, only 175 coarse time steps have been used to solve the homogenized equation which corresponds to a reduction of the complexity of the scheme by a factor of 35000 in the case of the coarse tessellation with 9 interior nodes. a and the map (F_1, F_2) are drawn in figure 23. L^1, L^2, L_∞ and H_1 errors are given in figure 24 to 27. Coarse and fine mesh errors are given in table 13 and 14 at time $t = 0.1$. We have chosen $g = 1$ in this numerical experiment, one obtains similar results by choosing $g = \sin(2.4x - 1.8y + 2\pi t)$.



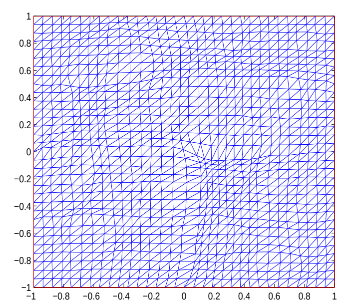
(a) a **at** $t = 0$**.**



(b) (F_1, F_2) **at** $t = 0$**.**

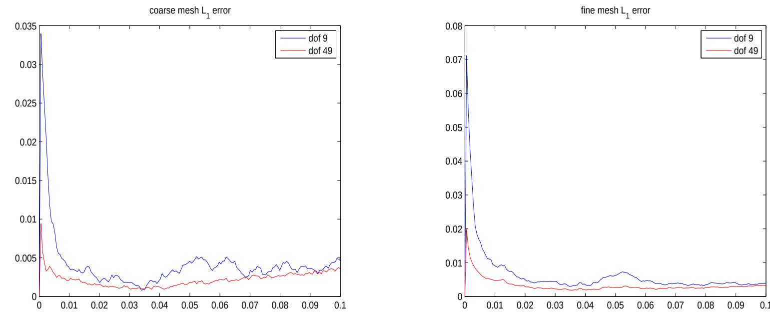


(c) a **at** $t = 0.1$**.**



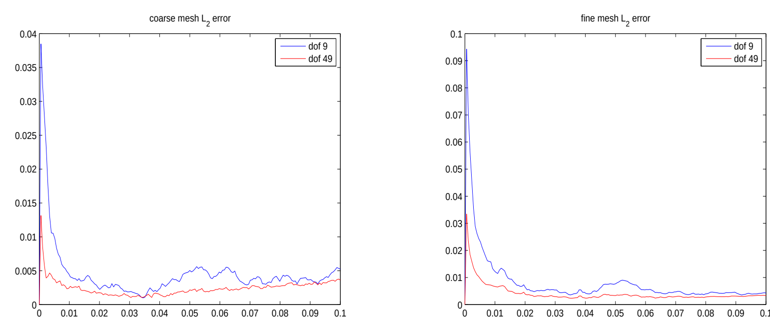
(d) (F_1, F_2) **at** $t = 0.1$**.**

Figure 23: a and (F_1, F_2) at time $t = 0$, $t = 0.1$ for the time dependent random fractal medium.



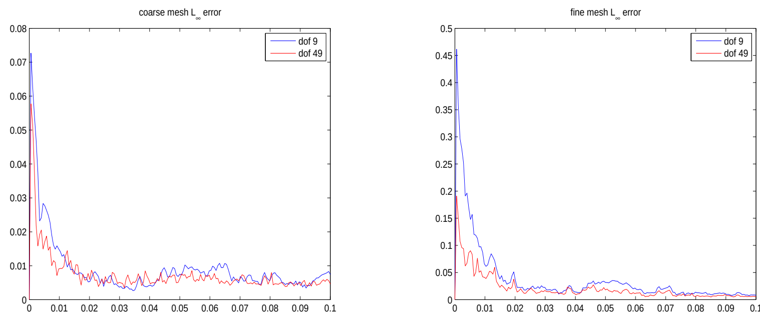
(a) ***coarse mesh*** L^1 ***error.*** (b) ***fine mesh*** L^1 ***error.***

Figure 24: L^1 error for the time dependent random fractal medium at $t = 0.1$.



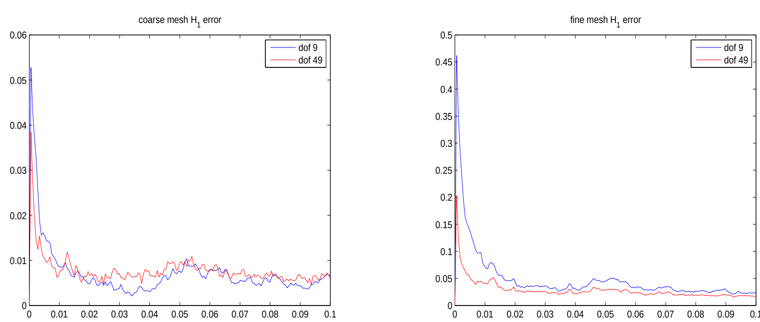
(a) ***coarse mesh*** L^2 ***error.*** (b) ***Fine Mesh*** L^2 ***error.***

Figure 25: L^2 error for the time dependent random fractal medium at $t = 0.1$..



(a) ***coarse mesh*** L_∞ ***error.*** **(b)** ***fine mesh*** L_∞ ***error.***

Figure 26: L^∞ error for the time dependent random fractal medium at $t = 0.1$..



(a) ***coarse Mesh*** $H1$ ***error.*** **(b)** ***fine Mesh*** H^1 ***error.***

Figure 27: H^1 error for the time dependent random fractal medium at $t = 0.1$..

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